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OPERATOR-VALUED KIRCHBERG THEORY
AND ITS CONNECTION TO TENSOR NORMS AND CORRESPONDENCES

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2015

Urbana, Illinois

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Abstract

In this thesis, we will first follow Kirchberg's categorical perspective to establish operator-valued WEP and QWEP. We develop similar properties as that in the classical WEP and QWEP, and illustrate the relations with the classical cases by some examples. Then we will discuss the notion of relative WEP in the context of Hilbert correspondence and investigate the relations between relatively weak injectivity and relative amenability. Finally we will apply our discoveries to recent results on C^* -norms, and generically find a mechanism to construct a continuum number of C^* -norms on some tensor products which admit infinitely many copies.

To Father and Mother.

Acknowledgments

This project would not have been possible without the support of many people. Many thanks to my adviser, Zhong-Jin Ruan, for his encouragement and helps, as well as many comments on my thesis. Many thanks to Professor Marius Junge, for having extensive inspiring discussions. Also thanks to my committee members, Florin Boca, and Ali Kavruk, who have offered guidance and support. Thanks to the Department of Mathematics, providing me with the financial means to complete this project. And finally, thanks to my wife, parents, and numerous friends who endured this long process with me, always offering support and love.

Table of Contents

List of Abbreviations	vi
List of Symbols	vii
Chapter 1 Introduction	1
Chapter 2 Preliminaries	4
2.1 Basics of C^* -algebras and von Neumann Algebras	4
2.1.1 Completely positive maps and Stinespring's theorem	5
2.1.2 Tensor products	5
2.1.3 Approximation properties and tensorial characterization	10
2.1.4 Type decomposition for von Neumann algebras	10
2.1.5 Universal Enveloping von Neumann Algebra	11
2.1.6 Ultrafilter and ultraproduct of C^* -algebras	13
2.2 WEP and QWEP	15
2.3 LP and LLP	17
2.4 The QWEP Conjecture and the Connes Embedding Problem	18
2.5 Hilbert C^* -Modules	18
2.5.1 Kirchberg's observations on the multiplier algebras	22
Chapter 3 Operator-valued Kirchberg Theory	24
3.1 Module version of the weak expectation property	25
3.2 Module version of QWEP	31
3.3 Illustrations	34
Chapter 4 Correspondence	38
4.1 Preliminaries	40
4.1.1 Correspondences	40
4.1.2 Local reflexivity	45
4.1.3 Basic construction and relative amenability	45
4.2 Proof of the main theorem	47
4.3 Relative Amenability	48
4.4 Relation with relatively weak injectivity	50
Chapter 5 Construction of Many Tensor Norms	59
5.1 Application to C^* -norms	60
5.2 Construction of norms from subalgebras	65
5.3 Constructions of norms from quotients	69
5.4 Constructions of norms from subalgebras and quotients	73
References	76

List of Abbreviations

u.c.p.	unital completely positive.
c.p.	completely positive.
c.c.p.	complete contractive positive
<i>r.w.i.</i>	relatively weakly injective.
WEP	the weak expectation property
QWEP	the quotient of a C^* -algebra with WEP
LLP	the local lifting property

List of Symbols

A , B , C and D C^* -algebras, unless specified.

$A \otimes B$ the algebraic tensor product of A and B .

M , N , and \mathcal{N} von Neumann algebras.

\mathbb{N} the set of the natural numbers.

\mathbb{R} the set of the real numbers.

\mathbb{C} the set of the complex numbers.

\mathcal{H} , \mathcal{K} Hilbert spaces.

$\mathbb{B}(\mathcal{H})$ the algebra of bounded operators on Hilbert space \mathcal{H} .

$\mathbb{K}(\mathcal{H})$, \mathbb{K} the algebra of compact operators on Hilbert space \mathcal{H} .

Γ discrete group.

\mathbb{F}_n free group of n generators.

\mathbb{F}_∞ free group of countably infinite generators.

$C_\lambda^*\Gamma$ the reduced group C^* -algebra of Γ .

$C^*\Gamma$ the full group C^* -algebra of Γ .

$\mathcal{L}(E, F)$ linear operators from space E to F .

\mathbb{M}_n $n \times n$ complex matrices.

$E_{i,j}$ standard matrix units in \mathbb{M}_n .

\mathcal{OS}_n n -dimensional operator spaces

Chapter 1

Introduction

In this thesis, we investigate a new notion of operator-valued WEP and QWEP. Let us recall that the weak expectation property (abbreviated as WEP) was introduced by E. Christopher Lance in his paper [Lan73] of 1973, as a generalization of injectivity of von Neumann algebras. In 1993, Eberhard Kirchberg [Ki93] revealed remarkable connections between tensor products of C^* -algebras and Lance's weak expectation property. He defined the notion of QWEP as a quotient of a C^* -algebra with the WEP, and formulated the famous QWEP conjecture that all C^* -algebras are QWEP. He showed a number of equivalences between various open problems in operator algebras. In particular, he showed that the QWEP conjecture is equivalent to an affirmative answer to the Connes Embedding Problem.

The motivation of our research is to generalize the notion of WEP and QWEP in the setting of Hilbert C^* -modules, in which the inner product of a Hilbert space is replaced by a C^* -valued inner product. Hilbert C^* -modules were first introduced in the work of Irving Kaplansky in 1953 [Kap53], in which he developed the theory for commutative, unital algebras. In the 1970s the theory was extended to noncommutative C^* -algebras independently by William Lindall Paschke [Pas73] and Marc Rieffel [Rie74]. The latter used Hilbert C^* -modules to construct a theory of induced representations of C^* -algebras. Hilbert C^* -modules are crucial to Kasparov's formulation of KK-theory [Kas80], and provide the right framework to extend the notion of Morita equivalence to C^* -algebras [Rie82]. They can be viewed as a generalization of vector bundles to noncommutative C^* -algebras and as such play an important role in noncommutative geometry, notably in C^* -algebraic quantum group theory and groupoid C^* -algebras.

Another motivation of our research is from the relation with amenable correspondences. The notion of correspondence of two von Neumann algebras has been introduced by Alain Connes and Vaughan Jones [CJ85], as a very useful tool for the study of type II_1 factors. Later Sorin Popa systematically developed this point of view to get some new insights in this area [Pop]. Among many interesting results and remarks, he discussed Connes' classical work on the injective II_1 factor in the framework of

correspondences, and he defined and studied a natural notion of amenability for a finite von Neumann algebra relative to a von Neumann subalgebra using conditional expectations. As Lance was inspired by Tomiyama's work on conditional expectations, we are interested in weak conditional expectations relative to a C^* -algebra.

Here is the plan of this thesis. Chapter 2 contains preliminaries on C^* -algebras, von Neumann algebras, tensor products of C^* -algebras and approximation properties. Moreover, we will list some useful results in classical WEP and QWEP theory, mostly by Kirchberg [Ki93]. We will omit most of the proofs.

Chapter 3 contains the main results on relative WEP and QWEP, which are inspired by Kirchberg's seminal work on non-semisplit extensions. We define two notions of WEP relative to a C^* -algebra D . Let E_D be a Hilbert D -module, and $\mathcal{L}(E_D)$ be the C^* -algebra of bounded adjointable linear operators on E_D . Also let $E_{D^{**}}$ be the weakly closed Hilbert D^{**} -module, and $\mathcal{L}^w(E_{D^{**}})$ be the von Neumann algebra of bounded adjointable linear operators on $E_{D^{**}}$. We say that a C^* -algebra A has the $DWEP_1$ if it is relatively weakly injective in $\mathcal{L}(E_D)$, i.e. for a faithful representation $A \subset \mathcal{L}(E_D)$, there exists a u.c.p. map $\mathcal{L}(E_D) \rightarrow A^{**}$, which preserves the identity on A . Respectively we define the $DWEP_2$ to be relatively weak injectivity in $\mathcal{L}^w(E_{D^{**}})$. We show that $DWEP_1$ implies $DWEP_2$, but the converse is not true. After investigating some basic properties, we establish a tensor product characterization of $DWEP$. Let \max_1^D be the tensor norm on $A \otimes C^*\mathbb{F}_\infty$ induced from the inclusion $A \otimes C^*\mathbb{F}_\infty \subseteq \mathcal{L}(E_D^u) \otimes_{\max} C^*\mathbb{F}_\infty$ for some universal Hilbert D -module E_D^u and $A \subset \mathcal{L}(E_D^u)$. Then a C^* -algebra A has the $DWEP_1$, if and only if

$$A \otimes_{\max_1^D} C^*\mathbb{F}_\infty = A \otimes_{\max} C^*\mathbb{F}_\infty.$$

We have the similar result for $DWEP_2$ with respect to some universal weakly closed D^{**} -module $E_{D^{**}}^u$.

Following the notion of relative WEP, we define two notions of the relative QWEP, derived from relative WEP. After developing basic properties of relative QWEP, we show that the two notions are equivalent, in contrast to the case of the relative WEP. Similarly, we establish a tensor product characterization of relative QWEP. Also we investigate some properties of WEP and QWEP relative to some special classes of C^* -algebras, and illustrate the relations with classical results in the theory of WEP and QWEP.

In Chapter 4, we examine the connections between relative weak injectivity and the weak contain-

ment of Hilbert correspondences. Given a pair of von Neumann algebras $N \subset M$, one finds many situations where there exists a norm one projection E from M onto N , and one may ask what properties of M are automatically inherited by N . This question is related to the concept of correspondence between two von Neumann algebras, as a very useful tool for the study of type II_1 factors.

The main problem of this chapter is the following. Suppose that a C^* -algebra A is represented faithfully in a von Neumann algebra M with N being the weak closure of A in M . If the inclusion $A \subset M$ is *r.w.i.*, can we find a conditional expectation from M to N ? This problem is related to the problem of whether the Hilbert $A - N$ correspondence ${}_A L_2(N)_N$ is weakly contained in ${}_A L_2(M)_N$. It turns out the problem has a negative answer. In particular, we find an example that $A \subset B \subset M$, in which A and B are C^* -algebras sitting in a von Neumann algebra M , such that $A \subset M$ is *r.w.i.*, but $B \subset M$ is not.

In Chapter 5, we discuss some applications of our tensor product characterization result for *DWEP* from Chapter 3 to construct more C^* -norms. As we know, the algebraic tensor product $A \otimes B$ of two C^* -algebras may admit distinct norms, for instance, the minimal and maximal norms. A C^* -algebra A such that $\|\cdot\|_{\min} = \|\cdot\|_{\max}$ on $A \otimes B$ for any other C^* -algebra B is called nuclear. In particular, Simon Wassermann [Was76] shows that $\mathbb{B}(\mathcal{H})$ is not nuclear, and later Gilles Pisier and Marius Junge [JP95] show that $\|\cdot\|_{\min} \neq \|\cdot\|_{\max}$ on $\mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\mathcal{H})$. Recently, Pisier and Ozawa [OP14] showed that there is at least a continuum of different C^* -norms on $\mathbb{B}(\ell_2) \otimes \mathbb{B}(\ell_2)$.

We adopt the idea in their paper to construct a new C^* -norm on $A \otimes B$ by using the notion of *DWEP* and the \max_1^D norm constructed in Chapter 3. We provide the conditions which make it neither min nor max norm, thus distinct from the continuum norms constructed by Pisier and Ozawa. We also give a concrete example satisfying the conditions and hence possessing four distinct tensor norms. These conditions will give us a new way to distinguish norms on C^* -algebras.

Moreover, we find a simple mechanism to construct a continuum number of distinct norms on tensor products, which admits infinite many copies. As a corollary, we find a new construction of a continuum number of distinct norms on $\mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\mathcal{H})$, which covers the result in [OP14]. Moreover, if we assume that the Connes embedding problem has a negative answer, then $C^*\mathbb{F}_\infty \otimes C^*\mathbb{F}_\infty$, not only have different $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ norms, but also admits 2^{\aleph_0} distinct C^* -norms.

In reality, this thesis consists of three rather disjoint parts. Results in Chapter 3 are from joint work with Sepideh Rezvani. Hence some of the proofs will be omitted here and will appear in Rezvani's thesis. Results in Chapter 4 and 5 are from the joint work with Marius Junge.

Chapter 2

Preliminaries

This chapter contains preliminaries which will be used throughout the thesis. The reader is assumed to be familiar with general functional analysis and basic C^* -algebra theory, and have a reasonably good knowledge of von Neumann algebras.

2.1 Basics of C^* -algebras and von Neumann Algebras

This section most serves the purpose of fixing the notation concerning the basic results in C^* -algebras and von Neumann algebras.

In this thesis, C^* -algebras are neither assumed to be unital nor separable unless explicitly stated. We will use the symbols A , B , C and D for C^* -algebras, and π for a $*$ -homomorphism. For given C^* -algebra A and a positive linear functional ϕ on A , we let $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ denote the GNS-construction corresponding to ϕ .

A von Neumann algebra is a self-adjoint algebra of bounded linear operators on a Hilbert space, which contains the identity and is closed in the weak operator topology. We will use the letters M and N to denote von Neumann algebras, and τ to denote traces. For a group Γ we will denote the group von Neumann algebra associated to Γ by $L\Gamma$.

We also recall that there are many interesting locally convex topologies on the set of bounded linear operators on a Hilbert space, and some relations between these topologies. Namely, the weak operator topology is weaker than both the strong operator topology and the ultraweak operator topology; the strong operator topology is weaker than the strong $*$ operator topology and the ultrastrong operator topology; the strong $*$ operator topology is weaker than the ultrastrong $*$ operator topology; the ultraweak operator topology is weaker than the ultrastrong operator topology, which is weaker than the ultrastrong $*$ operator topology. Also, all these topologies are weaker than the uniform topology, that is, the norm topology on $\mathbb{B}(\mathcal{H})$. See [Tak1] Chapter II for details.

2.1.1 Completely positive maps and Stinespring's theorem

Completely positive maps are the heart and soul of C^* -approximation theory.

Definition 2.1.1. *An operator system E is a closed self-adjoint subspace of a unital C^* -algebra A such that $1_A \in E$. For each $n \in \mathbb{N}$, let $\mathbb{M}_n(E)$ denote the matrix algebra over E , inherit an order structure from $\mathbb{M}_n(A)$: an element in $\mathbb{M}_n(E)$ is positive if and only if it is positive in $\mathbb{M}_n(A)$.*

A map φ from an operator system E to a C^* -algebra B is said to be *completely positive* if $\varphi_n : \mathbb{M}_n(E) \rightarrow \mathbb{M}_n(B)$, defined by

$$\varphi_n([a_{i,j}]) = [\varphi(a_{i,j})],$$

is positive for every n .

We use c.p. to abbreviate “completely positive”, u.c.p. for “unital completely positive”, and c.c.p. for “contractive completely positive”.

Directly generalizing the GNS construction, we have Stinespring's Representation Theorem for c.p. maps. The details of the proof can be found in many places (for example in [BrOz] Theorem 1.5.3), however we need the explicit construction and hence we reproduce the main ingredients.

Theorem 2.1.2 (Stinespring). *Let A be a unital C^* -algebra and $\varphi : A \rightarrow \mathbb{B}(\mathcal{H})$ be a c.p. map. Then there exist a Hilbert Space $\hat{\mathcal{H}}$, a $*$ -representation $\pi : A \rightarrow \mathbb{B}(\hat{\mathcal{H}})$, and an operator $V : \mathcal{H} \rightarrow \hat{\mathcal{H}}$ such that*

$$\varphi(a) = V^* \pi(a) V$$

for every $a \in A$. In particular, we can choose V such that $\|\varphi\| = \|V^* V\| = \|\varphi(1)\|$.

2.1.2 Tensor products

This section contains the necessary results on tensor products of C^* -algebras needed for this thesis. Most of the proofs are omitted, and the results can be found in the literature. For example, the results on the maximal and minimal tensor products are all contained in [BO08, Chapter 3]. Tensor products of operators are discussed in detail in [KR83, Section 2.6].

The readers are assumed to be familiar with the algebraic tensor product. We denote the algebraic tensor product of vector spaces V and W by $V \otimes W$, and elementary tensors in the algebraic tensor product are denoted by $v \otimes w$, for $v \in V$ and $w \in W$.

The readers are also expected to be familiar with the tensor product of Hilbert spaces and operators on Hilbert spaces. For Hilbert spaces \mathcal{H} and \mathcal{K} , we denote by $\mathcal{H} \otimes \mathcal{K}$ their tensor product. For bounded linear operators x and y on \mathcal{H} and \mathcal{K} , respectively, we denote by $x \otimes y$ the tensor product operator on $\mathcal{H} \otimes \mathcal{K}$. It is uniquely determined by acting on elementary tensors by $(x \otimes y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta)$, $\xi \in \mathcal{H}$, $\eta \in \mathcal{K}$. For a Hilbert space H , we denote by $H^{\otimes n}$ the n -fold tensor product $\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$.

If A , B and C are C^* -algebras, and $\pi_A : A \rightarrow C$ and $\pi_B : B \rightarrow C$ are $*$ -homomorphisms with commuting ranges, then we denote by $\pi_A \times \pi_B$ the $*$ -homomorphism

$$\pi_A \times \pi_B : A \otimes B \rightarrow C \quad \text{defined by} \quad (\pi_A \times \pi_B)(a \otimes b) = \pi_A(a)\pi_B(b),$$

for $a \in A$ and $b \in B$.

Proposition 2.1.3. *Given C^* -algebras A and B , a Hilbert space \mathcal{H} and a $*$ -homomorphism $\pi : A \otimes B \rightarrow \mathbb{B}(\mathcal{H})$, there exist $*$ -homomorphisms $\pi_A : A \rightarrow \mathbb{B}(\mathcal{H})$ and $\pi_B : B \rightarrow \mathbb{B}(\mathcal{H})$ with commuting ranges, such that $\pi = \pi_A \times \pi_B$.*

The maps π_A and π_B from the above proposition are called the restrictions of π .

Given C^* -algebras A and B together with representations $\pi_A : A \rightarrow \mathbb{B}(\mathcal{H})$ and $\pi_B : B \rightarrow \mathbb{B}(\mathcal{K})$ on Hilbert spaces \mathcal{H} and \mathcal{K} , respectively, we get a $*$ -representation

$$\pi_A \otimes \pi_B : A \otimes B \rightarrow \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$$

which is defined on elementary tensors by

$$(\pi_A \otimes \pi_B)(a \otimes b) = \pi_A(a) \otimes \pi_B(b).$$

One can define several C^* -norms on the algebraic tensor product of C^* -algebras. The most important are the maximal norm and the minimal norm, whose definitions we now recall.

Definition 2.1.4. *Given C^* -algebras A and B , the maximal tensor product of A and B is the completion of $A \otimes B$, with respect the norm*

$$\|x\|_{\max} = \sup\{\|\pi(x)\|, \quad \pi : A \otimes B \rightarrow \mathbb{B}(\mathcal{H}) \text{ is a } * \text{-representation}\},$$

for $x \in A \otimes B$, and it is denoted by $A \otimes_{\max} B$.

The maximal tensor norm is well-defined, and $A \otimes_{\max} B$ is a C^* -algebra. The maximal norm turns out to be the largest possible C^* -norm on $A \otimes B$.

Definition 2.1.5. *Given C^* -algebras A and B , the minimal tensor product of A and B is the completion of $A \otimes B$, with respect to the norm*

$$\left\| \sum_{k=1}^n a_k \otimes b_k \right\|_{\min} = \left\| \sum_{k=1}^n \pi(a_k) \otimes \rho(b_k) \right\|_{\mathbb{B}(\mathcal{H} \otimes \mathcal{K})},$$

for $a_1, \dots, a_n \in A$ and $b_1, \dots, b_n \in B$, and some choice of faithful representations $\pi : A \rightarrow \mathbb{B}(\mathcal{H})$ and $\rho : B \rightarrow \mathbb{B}(\mathcal{K})$. This completion is denoted by $A \otimes_{\min} B$.

The minimal tensor product is also called the spatial tensor product. It can be shown that \otimes_{\min} is independent of the choice of faithful representation. A famous theorem of Takesaki states that the minimal tensor norm is, in fact, the smallest C^* -norm on $A \otimes B$.

The maximal and minimal tensor product norms are both cross-norms, meaning that $\|x \otimes y\| = \|x\| \|y\|$ holds for all elementary tensors $x \otimes y$.

Since the maximal norm and the minimal norm are the largest and the smallest C^* -norms on $A \otimes B$ respectively, we obtain for any other C^* -norm $\|\cdot\|_{\alpha}$, canonical surjective $*$ -homomorphisms

$$A \otimes_{\max} B \rightarrow A \otimes_{\alpha} B \rightarrow A \otimes_{\min} B,$$

where $A \otimes_{\alpha} B$ denotes the completion of $A \otimes B$ with respect to the norm $\|\cdot\|_{\alpha}$. By canonical maps we mean that they restrict to the identity on the algebraic tensor product. We deduce that there is a unique C^* -norm on $A \otimes B$ if and only if $A \otimes_{\max} B = A \otimes_{\min} B$.

Later in the thesis, we will be interested in cases where $A \otimes_{\max} B = A \otimes_{\min} B$. A particular case where this happens, is if A or B is equal to \mathbb{M}_n , for some $n \in \mathbb{N}$.

The maximal tensor product has the following universal property:

Proposition 2.1.6. *Suppose that A , B and C are C^* -algebras. Given a $*$ -homomorphism $\pi : A \otimes B \rightarrow C$, there exists a unique map $A \otimes_{\max} B \rightarrow C$ extending π . In particular, if $\pi_A : A \rightarrow C$ and $\pi_B : B \rightarrow C$ are C^* -algebras with commuting ranges, then they induce a unique $*$ -homomorphism*

$$\pi_A \times \pi_B : A \otimes_{\max} B \rightarrow C.$$

Since a map $\pi_A \times \pi_B$ always extends from the algebraic tensor product to the maximal tensor

product, we will use the symbol $\pi_A \times \pi_B$ to mean both these things.

Since restrictions always exist, every $*$ -homomorphism π going out of the maximal tensor product has the form $\pi = \pi_A \times \pi_B$.

Now, let us turn our attention to maps between tensor products, and continuity properties of such.

Theorem 2.1.7. *Suppose that A_i and B_i , $i = 1, 2$, are C^* -algebras, together with completely positive maps $\varphi_i : A_i \rightarrow B_i$, $i = 1, 2$. Then the map*

$$A_1 \otimes A_2 \rightarrow B_1 \otimes B_2 \quad \text{given by} \quad a_1 \otimes a_2 \mapsto \varphi_1(a_1) \otimes \varphi_2(a_2)$$

extends to a completely positive map

$$\varphi_1 \otimes_{\max} \varphi_2 : A_1 \otimes_{\max} A_2 \rightarrow B_1 \otimes_{\max} B_2,$$

and it extends to a completely positive map

$$\varphi_1 \otimes_{\min} \varphi_2 : A_1 \otimes_{\min} A_2 \rightarrow B_1 \otimes_{\min} B_2.$$

Moreover, these satisfy $\|\varphi_1 \otimes_{\max} \varphi_2\| = \|\varphi_1 \otimes_{\min} \varphi_2\| = \|\varphi_1\| \|\varphi_2\|$

A particular case of the above theorem is when φ_1 and φ_2 are $*$ -homomorphisms. In this case, continuity ensures that both $\varphi_1 \otimes_{\max} \varphi_2$ and $\varphi_1 \otimes_{\min} \varphi_2$ are again $*$ -homomorphisms.

In the above theorem we used the notation $\varphi_1 \otimes_{\max} \varphi_2$ and $\varphi_1 \otimes_{\min} \varphi_2$, for these specific tensor product maps, but later we will use $\varphi_1 \otimes \varphi_2$ as a generic symbol for most tensor product maps. It should be clear from the context which maps we are talking about.

At this point, let us make some comments on a particular class of maps, namely, the inclusion of a C^* -subalgebra into a C^* -algebra. The following proposition follows directly from the fact that the minimal tensor norm is independent of the choice of faithful representation.

Proposition 2.1.8. *Given C^* -algebras B_i and C^* -subalgebras $A_i \subseteq B_i$, $i = 1, 2$, the minimal tensor norm on $B_1 \otimes B_2$ restricts to the minimal tensor norm on $A_1 \otimes A_2$. Hence, the inclusion of the algebraic tensor products induces an isometric inclusion*

$$A_1 \otimes_{\min} A_2 \subseteq B_1 \otimes_{\min} B_2.$$

There is no analogue of the above proposition for the maximal tensor product. There always exists a map

$$A_1 \otimes_{\max} A_2 \rightarrow B_1 \otimes_{\max} B_2,$$

and it maps surjectively onto the closure of the algebraic tensor product $A_1 \otimes A_2$ inside $B_1 \otimes B_2$, with respect to the maximal tensor norm on $B_1 \otimes B_2$, but we are not guaranteed that it is injective. We will see later that for a C^* -algebra B and a C^* -subalgebra $A \subseteq B$, the inclusion $A \otimes_{\max} C \subseteq B \otimes_{\max} C$ being isometric for all C^* -algebras C is equivalent to A being *relatively weakly injective* in B .

One would also like to know how the tensor products behave with respect to exact sequences. Unlike the case of inclusions, in this case there is an easy answer for the maximal tensor product.

Proposition 2.1.9. *Given a C^* -algebra A and an ideal I in A , the sequence*

$$0 \rightarrow I \otimes_{\max} B \rightarrow A \otimes_{\max} B \rightarrow (A/I) \otimes_{\max} B \rightarrow 0$$

is exact for all C^ -algebras B , where all the maps are the obvious ones.*

However, the answer for the minimal tensor product is not necessarily true. In this case, among other results we have the following proposition.

Proposition 2.1.10. *Suppose that A and B are C^* -algebras, and that I is an ideal in A . If there is a unique norm on $(A/I) \otimes B$, then the sequence*

$$0 \rightarrow I \otimes_{\min} B \rightarrow A \otimes_{\min} B \rightarrow (A/I) \otimes_{\min} B \rightarrow 0$$

is exact, where all the maps are the obvious ones.

Before ending this section we will talk about some other tensor products, namely the von Neumann algebra tensor product.

Suppose that we are given von Neumann algebras M and N on Hilbert spaces \mathcal{H} and \mathcal{K} . The von Neumann algebra tensor product of M and N , denoted by $M \bar{\otimes} N$, is the set

$$M \bar{\otimes} N = \{x \otimes y : x \in M, y \in N\}'' ,$$

which is also the strong operator closure of the set $\{x \otimes y : x \in M, y \in N\}$ in $\mathbb{B}(\mathcal{H} \otimes \mathcal{K})$. For a von Neumann algebra M we will denote by $M^{\bar{\otimes} n}$ the n -fold tensor product $M \bar{\otimes} M \bar{\otimes} \cdots \bar{\otimes} M$.

2.1.3 Approximation properties and tensorial characterization

Nuclearity and Exactness have dominated the C^* -scene for quite a while. We will give a short review on these notions, and especially review the tensorial characterization.

Definition 2.1.11. A map $\theta : A \rightarrow B$ is called *nuclear* if there exist c.c.p. maps $\varphi_n : A \rightarrow \mathbb{M}_{k(n)}(\mathbb{C})$, and $\psi : \mathbb{M}_{k(n)}(\mathbb{C}) \rightarrow B$ such that $\psi_n \circ \varphi_n \rightarrow \theta$ in the point-norm topology:

$$\|\psi_n \circ \varphi_n(a) - \theta(a)\| \rightarrow 0,$$

for all $a \in A$.

A C^* -algebra A is called *nuclear* if the identity map $\text{id} : A \rightarrow A$ is nuclear. A is called *exact*, if there exists a faithful representation $\pi : A \rightarrow \mathbb{B}(\mathcal{H})$ such that π is nuclear.

It is obvious that nuclearity of A implies exactness. Some examples of nuclear C^* -algebras are abelian C^* -algebras, finite dimensional C^* -algebras, and the group C^* -algebra of an amenable group.

The most important characterization of nuclearity and exactness are the following.

Theorem 2.1.12. For a C^* -algebra A , we have that

1. (Choi and Effros) A is nuclear, if and only if for every C^* -algebra B , there exists a unique C^* -norm on $A \otimes B$.
2. (Kirchberg) A is exact, if and only if for each C^* -algebra B and ideal J in B , the sequence

$$0 \rightarrow J \otimes_{\min} A \rightarrow B \otimes_{\min} A \rightarrow (B/J) \otimes_{\min} A \rightarrow 0$$

is exact.

2.1.4 Type decomposition for von Neumann algebras

Let us recall the type decomposition of a von Neumann algebra. We start by reviewing the concepts of abelian, finite and infinite projections.

Definition 2.1.13. Let M be a von Neumann algebra, and $p \in M$ a projection. We say that p is *abelian* if pMp is abelian, and we say that p is *finite* if whenever $q \in M$ is a projection equivalent to p with $q \leq p$, then $q = p$. A projection which is not finite is called *infinite*. The von Neumann algebra

M is called *finite*, if the identity in M is finite, and *properly infinite*, if it does not contain any central non-zero finite projections.

We now define the type of a von Neumann algebra.

Definition 2.1.14. *Let M be a von Neumann algebra. We say that M is: of type I if every non-zero projection majorizes a non-zero abelian projection; of type II if it does not contain any non-zero abelian projection and every non-zero projection majorizes a non-zero finite projection; of type III if it does not contain any non-zero finite projections.*

The type decomposition then says the following:

Proposition 2.1.15. *Every von Neumann algebra M can be written uniquely as a direct sum $M_I \oplus M_{II} \oplus M_{III}$, where M_I , M_{II} and M_{III} are either zero or of type I, II and III, respectively. Also, every von Neumann algebra M can be written uniquely as a direct sum $M_f \oplus M_\infty$, where M_f is finite or zero and M_∞ is properly infinite or zero.*

Combining the two statements above one obtains, that every von Neumann algebra can be written uniquely as a direct sum of five kinds of von Neumann algebras, finite of type I, properly infinite of type I, finite of type II, properly infinite of type II, and type III, respectively.

By a von Neumann algebra factor we shall understand a von Neumann algebra, whose center consists only of scalar multiples of the identity. Clearly, a factor is exactly of one of the three types mentioned. It is also either finite or properly infinite. Finite von Neumann algebras of type II are called type II_1 von Neumann algebras. Recall also that if M is a finite von Neumann algebra factor, then either M is isomorphic to M_n for some $n \in \mathbb{N}$, in which case we say that M is of type I_n , or it is of type II_1 . The latter happens if and only if M has infinite linear dimension.

We will frequently use the fact that a finite von Neumann algebra factor has a unique faithful normal trace, see [KR83], Proposition 8.5.3.

2.1.5 Universal Enveloping von Neumann Algebra

We know that a von Neumann algebra can be thought of as a dual space in a particularly nice way. In this section, we shall see that the double dual of a C^* -algebra can be given the structure of a von Neumann algebra in a natural way.

We first introduce the notion of a universal representation.

Definition 2.1.16. Suppose that A is a C^* -algebra and $\pi : A \rightarrow \mathbb{B}(\mathcal{H})$ a representation of A on some Hilbert space \mathcal{H} . The representation π is called *universal* if π is non-degenerate, and satisfies the following universal property: given another non-degenerate representation ρ of A on some Hilbert space \mathcal{K} , there exists a surjective $*$ -homomorphism $\tilde{\rho} : \pi(A)'' \rightarrow \rho(A)''$, which is ultraweakly operator-to-ultraweak operator continuous, such that $\tilde{\rho} \circ \pi = \rho$. In other words, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\rho} & \rho(A)'' \\ \pi \downarrow & \nearrow \tilde{\rho} & \\ \pi(A)'' & & \end{array}$$

Now, the following proposition is the key ingredient in proving that the double dual of a C^* -algebra has a natural structure as a von Neumann algebra.

Proposition 2.1.17. Suppose that A is a C^* -algebra and $\pi : A \rightarrow \mathbb{B}(\mathcal{H})$ a non-degenerate representation of A on a Hilbert space \mathcal{H} . Then π extends uniquely to a weak $*$ -to-ultraweak continuous map $\pi : A^{**} \rightarrow \pi(A)''$, that is, π is weak $*$ -to-ultraweakly continuous and the diagram

$$\begin{array}{ccc} A & \xrightarrow{\pi} & \pi(A)'' \\ \iota \downarrow & \nearrow \tilde{\pi} & \\ A^{**} & & \end{array}$$

commutes. Here ι denotes the natural inclusion. Moreover, π maps the closed unit ball of A^{**} onto the closed unit ball of $\pi(A)''$, so in particular it is surjective.

Now we are ready to show that universal representations exist, and give the connection to the double dual.

Suppose that A is a C^* -algebra, and for each element φ in the state space $S(A)$ of A , let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be the GNS-construction corresponding to φ . We let π_u denote the direct sum of all these representations. So $\pi_u = \bigoplus_{\varphi \in S(A)} \pi_\varphi$, and it is a representation on the Hilbert space $\mathcal{H}_u = \bigoplus_{\varphi \in S(A)} \mathcal{H}_\varphi$. This representation is non-degenerate since each representation π_φ , $\varphi \in S(A)$ is so. It is well-known that this representation is faithful, since the states on A separate points. This representation is actually universal, and that $\pi_u(A)''$ can be identified with A^{**} in a natural way.

Theorem 2.1.18. Let A be a C^* -algebra. With the notation above, the representation π_u is a universal representation of A . Moreover, π_u extends to a surjective isometry $\tilde{\pi}_u : A^{**} \rightarrow \pi_u(A)''$, which is also

a weak*-to-ultraweak operator topology homeomorphism.

The von Neumann algebra $\pi_u(A)''$ from Theorem 2.1.18 is called the universal enveloping von Neumann algebra of A . The theorem justifies that we may identify A^{**} with this universal enveloping von Neumann algebra of A , and thus in the following we do not distinguish between the two spaces. It is also referred to as the double dual. Since a Banach space is weak*-dense in its double dual, we also get from Theorem 2.1.18 that A is ultraweakly dense in A^{**} .

Before we end this section, let us recall the following small result, which we will need a couple of times.

Proposition 2.1.19. *Suppose that A and B are C^* -algebras, and $\pi : B \rightarrow A$ is a surjective *-homomorphism. Let $I = \ker \pi$, and let p denote the central projection in B^{**} so that $pB^{**} = I^{**}$. If $\varphi : B^{**} \rightarrow I^{**}$ denotes multiplication by p , then the map $\varphi \oplus \pi^{**} : B^{**} \rightarrow I^{**} \oplus A^{**}$ is an isomorphism.*

2.1.6 Ultrafilter and ultraproduct of C^* -algebras

This section is a very short introduction to filters and ultraproduct. The purpose of this section is to set the terminology and state a number of results on filters that will be used frequently throughout the thesis.

Definition 2.1.20. *Suppose that I is an index set. A family \mathcal{F} of subsets of I is called a filter, if it satisfies the following three conditions:*

1. (nontriviality) the empty set is not in \mathcal{F} ;
2. (directedness) if $A \in \mathcal{F}$ and $B \subseteq I$ with $A \subseteq B$, then $B \in \mathcal{F}$;
3. (finite intersection property) if $A, B \in \mathcal{F}$, then their intersection is also in \mathcal{F} .

If in addition to the conditions above, the set \mathcal{F} satisfies:

1. (maximality) for each $A \subseteq I$, either $A \in \mathcal{F}$ or $I \setminus A \in \mathcal{F}$,

then \mathcal{F} is called an ultrafilter.

If I is a set and J a collection of subsets of I such that J has the finite intersection property, then there is a filter containing J , namely the set of all subsets I_0 of I such that there exist $I_1, \dots, I_n \in J$ with $I_1 \cap \dots \cap I_n \subseteq I_0$. This is called the filter generated by J .

It is straightforward to check that for any non-empty set $A \subset I$, the set

$$\mathcal{F} = \{B \subset I : A \subset B\}$$

is a filter on I . Such a filter is called a *principal* filter on I . An ultrafilter which is not principal is called *free* (or non-principal). If \mathcal{F} is a principal ultrafilter, then A must necessarily be a singleton, that is, A only has one point.

The notion of filter comes from topology. It can be used to axiomatize topological spaces. Suppose that X is a topological space, I is an index set and \mathcal{F} a filter on I . An indexed family $(x_i)_{i \in I}$ of elements in X is said to converge along the filter \mathcal{F} to some $x \in X$ if

$$\{i \in I : x_i \in U\} \in \mathcal{F}$$

for all open neighborhoods U of x . This is written $\lim_{i \rightarrow \mathcal{F}} x_i = x$.

It is straightforward to check that if the topological space is Hausdorff, then a potential limit along a filter is unique. This follows from the fact that a filter cannot contain the empty set.

The following theorem is probably the main reason that we, in this thesis, prefer ultrafilters, in contrast to just filters. For details of the proof, see [CSC10], Appendix J.

Theorem 2.1.21. *Suppose that X is a compact topological space, I an index set and ω an ultrafilter on I . Then every subset of X indexed by I converges along ω , that is, for every indexed subset $(x_i)_{i \in I}$ of X the limit $\lim_{i \rightarrow \omega} x_i$ exists.*

Let \mathcal{U} be an ultrafilter on a set I . Let $(X_i)_{i \in I}$ be a net of Banach spaces. We denote by $\Pi_{i \in I} X_i$ the ℓ_∞ -direct sum of the X_i 's and let $N_{\mathcal{U}}$ be the Banach space of \mathcal{U} -null nets:

$$N_{\mathcal{U}} = \{(x_i)_{i \in I} \in \Pi_{i \in I} X_i : \lim_{i \rightarrow \mathcal{U}} \|x_i\| = 0\}.$$

The *ultraproduct Banach space* of $(X_i)_{i \in I}$ is defined as $X_{\mathcal{U}} = (\Pi X_i) / N_{\mathcal{U}}$. We write $x_{\mathcal{U}}$ for the element represented by $(x_i)_{i \in I}$. It is easy to check that $\|x_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|$. If $A_i = X_i$ are all C*-algebras, then the ultraproduct $A_{\mathcal{U}}$ is again a C*-algebra. If $\mathcal{H}_i = X_i$ are all Hilbert spaces, then the ultraproduct $\mathcal{H}_{\mathcal{U}}$ is again a Hilbert space such that $\langle \eta_{\mathcal{U}}, \xi_{\mathcal{U}} \rangle = \lim_{\mathcal{U}} \langle \eta_i, \xi_i \rangle$. If $A_i \subset \mathbb{B}(\mathcal{H}_i)$, then $A_{\mathcal{U}} \subset \mathbb{B}(\mathcal{H}_{\mathcal{U}})$ with $a_{\mathcal{U}} \xi_{\mathcal{U}} = (a_i \xi_i)_{i \rightarrow \mathcal{U}}$.

Ultraproducts of von Neumann algebras are not quite as straightforward as the C*-case. To keep

the notation consistent, let ω be a free ultrafilter on \mathbb{N} . Let (M, τ) be a von Neumann algebra with a faithful tracial state τ . Let N_ω be the norm closed ideal of ΠM , given by

$$N_\omega = \{(x_n)_{n \in \mathbb{N}} \in \Pi M : \lim_{n \rightarrow \omega} \|x_n\|_2 = 0\},$$

where $\|x_n\|_2 = \tau(x_n^* x_n)^{1/2}$. The (tracial) ultraproduct of (M, τ) is defined to be $M^\omega = (\Pi M)/N_\omega$. It has a faithful tracial state τ_ω given by $\tau_\omega(x_\omega) = \lim_\omega \tau(x_n)$. We note that $\|x_\omega\|_2 := \tau_\omega(x_\omega^* x_\omega)^{1/2} = \lim_{n \rightarrow \omega} \|x_n\|_2$. It can be shown that M^ω is a von Neumann algebra and faithful tracial state τ_ω is normal. See [BrOz] Appendix A for details.

2.2 WEP and QWEP

The notion of WEP is from Lance [Lan73], inspired by Tomiyama's extensive work on conditional expectations. Kirchberg in [Ki93] raises the famous QWEP conjecture and establishes its several equivalences. Here we list some useful results for readers' convenience. Most of the results and proofs can be found in Ozawa's survey paper [Oz04].

Definition 2.2.1. *Suppose that B is a C^* -algebra and A is a C^* -subalgebra of B . We say A is relatively weakly injective (abbreviated as r.w.i.) in B , if there is a c.c.p. map $\varphi : B \rightarrow A^{**}$ such that $\varphi|_A = \text{id}_A$.*

For von Neumann algebras $N \subset M$, relative weak injectivity is equivalent to the existence of a (non-normal) conditional expectation from M to N .

The next proposition gives equivalent characterizations of relatively weakly injectivity.

Proposition 2.2.2. *Suppose that B is a C^* -algebra and A is a C^* -subalgebra of B . Then the following are equivalent:*

1. *A is relatively weakly injective in B ;*
2. *there exists a conditional expectation $\psi : B^{**} \rightarrow A^{**}$;*
3. *for every finite dimensional subspace $E \subseteq B$ and any $\varepsilon > 0$, there exist a linear contraction $\psi : E \rightarrow A$ such that $\|\psi|_{A \cap M} - \text{id}_{A \cap M}\| < \varepsilon$.*

We say a C^* -algebra A has the weak expectation property (short as WEP), if it is relatively weakly injective in $\mathbb{B}(\mathcal{H})$ for a faithful representation $A \subset \mathbb{B}(\mathcal{H})$.

Since $\mathbb{B}(\mathcal{H})$ is injective, the notion of WEP does not depend on the choice of a faithful representation of A . We say a C^* -algebra is QWEP if it is a quotient of a C^* -algebra with the WEP. The QWEP conjecture raised by Kirchberg in [Ki93] states that all C^* -algebras are QWEP.

From the definition of *r.w.i.*, it is easy to see the following transitivity property.

Lemma 2.2.3. *For C^* -algebras $A_0 \subseteq A_1 \subseteq A$, such that A_0 is relatively weakly injective in A_1 , A_1 is relatively weakly injective in A , then A_0 is relatively weakly injective in A .*

The property of *r.w.i.* is also closed under direct product.

Lemma 2.2.4. *If $(A_i)_{i \in I}$ is a net of C^* -algebras such that A_i is relatively weakly injective in B_i for all $i \in I$, then $\Pi_{i \in I} A_i$ is relatively weakly injective in $\Pi_{i \in I} B_i$.*

In [Lan73], Lance establishes the following tensor product characterization of the WEP. The proof of the theorem is called *The Trick*, and we will be using this throughout the paper. In the following, let \mathbb{F}_∞ denote the free group with countably infinite many generators, and $C^*\mathbb{F}_\infty$ be the full group C^* -algebra of \mathbb{F}_∞ .

Theorem 2.2.5. *Suppose that B is a C^* -algebra and A a C^* -subalgebra of B . Then the following are equivalent:*

1. *A is relatively weakly injective in B ;*
2. *for each representation $\pi : A \rightarrow \mathbb{B}(\mathcal{H})$ of A on a Hilbert space \mathcal{H} , there exists a contractive completely positive map $\varphi : B \rightarrow \pi(A)$ extending π ;*
3. *the inclusion $A \otimes_{\max} C \rightarrow B \otimes_{\max} C$ is isometric for every C^* -algebra C ;*
4. *the inclusion $A \otimes_{\max} C^*\mathbb{F}_\infty \rightarrow B \otimes_{\max} C^*\mathbb{F}_\infty$ is isometric.*

As a consequence of the above theorem, we have the following result.

Corollary 2.2.6. *A C^* -algebra A has the WEP if and only if for any inclusion $A \subseteq B$, A is relatively weakly injective in B .*

Similar to the WEP, the QWEP is also preserved by the relatively weak injectivity as following.

Lemma 2.2.7. *If a C^* -algebra A is relatively weakly injective in a QWEP C^* -algebra, then it is QWEP.*

Although the WEP does not pass to the double dual, the QWEP is more flexible.

Proposition 2.2.8. *A C^* -algebra A is QWEP if and only if A^{**} is QWEP.*

As a corollary of the above proposition, $\mathbb{B}(\mathcal{H})^{**}$ is QWEP. Moreover we have the following equivalence.

Corollary 2.2.9. *A C^* -algebra A is QWEP if and only if A is relatively weakly injective in $\mathbb{B}(\mathcal{H})^{**}$.*

2.3 LP and LLP

Let A be a C^* -algebra, J be a closed two-sided ideal in a C^* -algebra B , and $\pi : B \rightarrow B/J$ be the quotient map. We say a c.c.p. map $\varphi : A \rightarrow B/J$ is *liftable* if there exists a c.c.p. map $\psi : A \rightarrow B$ such that $\pi \circ \psi = \varphi$. We say φ is *locally liftable* if for any finite-dimensional operator system $E \subset A$, there exists a c.c.p. map $\psi : E \rightarrow B$ such that $\pi \circ \psi = \varphi|_E$.

Definition 2.3.1. *A unital C^* -algebra has the lifting property (LP) (resp. local lifting property (LLP)) if any c.c.p. map from A into a quotient C^* -algebra B/J is liftable (resp. locally liftable). A nonunital C^* -algebra has the LP (resp. LLP) if its unitization has the property.*

The Choi-Effros Lifting theorem implies that separable nuclear C^* -algebras have the LP. Kirchberg shows in [Ki93] that the full C^* -algebra $C^*\mathbb{F}_\infty$ of a countable free group \mathbb{F}_∞ has the LP. Moreover, he gives a tensorial characterization for the LLP and WEP, with the help of the following striking theorem.

Theorem 2.3.2 (Kirchberg [Ki93]). *For any free group \mathbb{F} and any Hilbert space \mathcal{H} , we have the isometric isomorphism*

$$C^*\mathbb{F} \otimes_{\max} \mathbb{B}(\mathcal{H}) = C^*\mathbb{F} \otimes_{\min} \mathbb{B}(\mathcal{H}).$$

The C^* -algebra $\mathbb{B}(\ell_2)$ is the home space in the sense that it contains all separable C^* -algebras. It has the WEP, since it is injective. The full free group C^* -algebra $C^*\mathbb{F}_\infty$ has the LP and any separable unital C^* -algebra is a quotient of it. With these simple observations, we can now establish the tensorial characterization of the WEP and LLP.

Corollary 2.3.3 (Kirchberg [Ki93]). *For C^* -algebras A and B , we have the following:*

1. $A \otimes_{\max} B = A \otimes_{\min} B$ canonically if A has the LLP and B has the WEP;

2. $A \otimes_{\max} \mathbb{B}(\mathcal{H}) = A \otimes_{\min} \mathbb{B}(\mathcal{H})$ canonically if and only if A has the LLP.
3. $C^*\mathbb{F}_{\infty} \otimes_{\max} B = C^*\mathbb{F}_{\infty} \otimes_{\min} B$ if and only if B has the WEP.

2.4 The QWEP Conjecture and the Connes Embedding Problem

Recall that a C^* -algebra A is *QWEP* if it is a quotient of C^* -algebra with the WEP. Kirchberg's QWEP conjecture asserts that every C^* -algebra is QWEP, and it turns out to be equivalent to several seemingly unrelated open problems, including the Connes embedding problem. Consequently, the QWEP conjecture is one of the most important open problems in the theory of operator algebras.

Theorem 2.4.1 (Kirchberg [Ki93]). *The following statements are equivalent:*

1. every C^* -algebra is QWEP;
2. $C^*\mathbb{F}_{\infty} \otimes_{\max} C^*\mathbb{F}_{\infty} = C^*\mathbb{F}_{\infty} \otimes_{\min} C^*\mathbb{F}_{\infty}$ canonically;
3. every type II_1 -factor with separable predual is embeddable into the ultraproduct R^{ω} of the hyperfinite type II_1 factor R ;
4. the (separable) predual of any von Neumann-algebra is isometrically isomorphic to a subspace of the Banach space ultraproduct $(S_1)_{\omega}$ of the predual S_1 of $\mathbb{B}(\ell_2)$.

2.5 Hilbert C^* -Modules

The notion of Hilbert C^* -modules first appeared in a paper by Irving Kaplansky [Kap53] in 1953. The theory was then developed by the work of William Lindall Paschke in [Pas73]. In this section we give a brief introduction to Hilbert C^* -modules and present some of their fundamental properties which we are going to use throughout this paper.

Definition 2.5.1. *Let D be a C^* -algebra. An inner-product D -module is a linear space E which is a right D -module with compatible scalar multiplication: $\lambda(xa) = (\lambda x)a = x(\lambda a)$, for $x \in E$, $a \in D$, $\lambda \in \mathbb{C}$, and a map $(x, y) \mapsto \langle x, y \rangle : E \times E \rightarrow D$ with the following properties:*

1. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for $x, y, z \in E$ and $\alpha, \beta \in \mathbb{C}$;

$$2. \langle x, ya \rangle = \langle x, y \rangle a \quad \text{for } x, y \in E \text{ and } a \in D;$$

$$3. \langle y, x \rangle = \langle x, y \rangle^* \quad \text{for } x, y \in E;$$

$$4. \langle x, x \rangle \geq 0; \text{ if } \langle x, x \rangle = 0, \text{ then } x = 0.$$

For $x \in E$, we let $\|x\| = \|\langle x, x \rangle\|^{1/2}$. It is easy to check that if E is an inner-product D -module, then $\|\cdot\|$ is a norm on E .

Definition 2.5.2. An inner-product D -module which is complete with respect to its norm is called a Hilbert D -module or a Hilbert C^* -module over the C^* -algebra D .

Note that any C^* -algebra D is a Hilbert D -module itself with the inner product $\langle x, y \rangle = x^*y$ for x and y in D . Another important example of a Hilbert C^* -module is the following.

Example 2.5.3. Let \mathcal{H} be a Hilbert space. Then the algebraic tensor product $\mathcal{H} \otimes_{alg} D$ can be equipped with a D -valued inner-product:

$$\langle \xi \otimes a, \eta \otimes b \rangle = \langle \xi, \eta \rangle a^*b \quad (\xi, \eta \in \mathcal{H}, a, b \in D).$$

Let $\mathcal{H}_D = \mathcal{H} \otimes D$ be the completion of $\mathcal{H} \otimes_{alg} D$ with respect to the induced norm. Then \mathcal{H}_D is a Hilbert D -module.

Let E and F be Hilbert D -modules. Let t be an adjointable map from E to F , i.e. there exists a map t^* from F to E such that

$$\langle tx, y \rangle = \langle x, t^*y \rangle, \quad \text{for } x \in E \text{ and } y \in F.$$

One can easily see that t must be right D -linear, that is, t is linear and $t(xa) = t(x)a$ for all $x \in E$ and $a \in D$. It follows that any adjointable map is bounded, but the converse is not true – a bounded D -linear map need not be adjointable. Let $\mathcal{L}(E, F)$ be the set of all adjointable maps from E to F , and we abbreviate $\mathcal{L}(E, E)$ to $\mathcal{L}(E)$. Note that $\mathcal{L}(E)$ is a C^* -algebra equipped with the operator norm.

Now we review the notion of compact operators on Hilbert D -modules, as an analogue to the compact operators on a Hilbert space. Let E and F be Hilbert D -modules. For every x in E and y

in F , define the map $\theta_{x,y} : E \rightarrow F$ by

$$\theta_{x,y}(z) = y\langle x, z \rangle \quad \text{for } z \in E.$$

One can check that $\theta_{x,y} \in \mathcal{L}(E, F)$ and $\theta_{x,y}^* = \theta_{y,x}$. We denote by $\mathbb{K}(E, F)$ the closed linear subspace of $\mathcal{L}(E, F)$ spanned by $\{\theta_{x,y} : x \in E, y \in F\}$, and we abbreviate $\mathbb{K}(E, E)$ to $\mathbb{K}(E)$. We call the elements of $\mathbb{K}(E, F)$ *compact operators*.

Let E be a Hilbert D -module and Z be a subset of E . We say that Z is a *generating set* for E if the closed submodule of E generated by Z is the whole of E . If E has a countable generating set, we say that E is countably generated.

In [Kas80], Kasparov proves the following theorem known as the *absorption theorem*, which shows the universality of \mathcal{H}_D in the category of Hilbert D -modules.

Theorem 2.5.4. *Let D be a C^* -algebra and E be a countably generated Hilbert D -module. Then $E \oplus \mathcal{H}_D \approx \mathcal{H}_D$, i.e. there exists an element $u \in \mathcal{L}(E \oplus \mathcal{H}_D, \mathcal{H}_D)$ such that $u^*u = 1_{E \oplus \mathcal{H}_D}$ and $uu^* = 1_{\mathcal{H}_D}$.*

Remark 2.5.5. Using the absorption theorem, for an arbitrary Hilbert D -module E , we have $\mathcal{L}(E \oplus \mathcal{H}_D) \simeq \mathcal{L}(\mathcal{H}_D)$. Hence we have an embedding of $\mathcal{L}(E)$ in $\mathcal{L}(\mathcal{H}_D)$ and a conditional expectation from $\mathcal{L}(\mathcal{H}_D)$ to $\mathcal{L}(E)$.

Before we proceed to the main results of Hilbert C^* -modules, let us recall the notion of *multiplier algebra* of a C^* -algebra.

Definition 2.5.6. *Let A and B be C^* -algebras. If A is an ideal in B , we call A an *essential ideal* if there is no non-zero ideal of B that has zero intersection with A . Or equivalently if $b \in B$ and $bA = \{0\}$, then $b = 0$.*

It can be shown that for any C^* -algebra A , there is a unique (up to isomorphism) maximal C^* -algebra which contains A as an essential ideal. This algebra is called the multiplier algebra of A and is denoted by $\mathcal{M}(A)$.

Theorem 2.5.7. *If E is a Hilbert D -module, then $\mathcal{L}(E) = \mathcal{M}(\mathcal{K}(E))$.*

Note that if $E = D$ for a unital C^* -algebra D , then $D = \mathbb{K}(D)$ and $\mathcal{L}(D) = \mathcal{M}(D)$.

In the special case where $E = \mathcal{H}_D$, we have

$$\mathcal{K}(\mathcal{H}_D) \simeq \mathcal{K}(\mathcal{H}) \otimes_{\min} D = \mathbb{K} \otimes D,$$

where $\mathbb{K} = \mathcal{K}(\mathcal{H})$ is the C^* -algebra of the compact operators. Therefore, by Theorem 2.5.7 we have

$$\mathcal{L}(\mathcal{H}_D) \simeq \mathcal{M}(\mathbb{K} \otimes_{\min} D).$$

In [Kas80] Kasparov introduces a GNS type of construction in the context of Hilbert C^* -modules, known as the KSGNS construction (for Kasparov, Stinespring, Gelfand, Neimark, Segal) as follows.

Theorem 2.5.8. *Let A be a C^* -algebra, E be a Hilbert D -module and let $\rho : A \rightarrow \mathcal{L}(E)$ be a completely positive map. There exists a Hilbert D -module E_ρ , a $*$ -homomorphism $\pi_\rho : A \rightarrow \mathcal{L}(E_\rho)$ and an element v_ρ of $\mathcal{L}(E, E_\rho)$, such that*

$$\rho(a) = v_\rho^* \pi_\rho(a) v_\rho \quad (a \in A),$$

$$\text{Span}\{\pi_\rho(A) v_\rho E\} \text{ is dense in } E_\rho.$$

As a consequence of the above theorem, Kasparov shows that given a C^* -algebra D , any separable C^* -algebra can be considered as a C^* -subalgebra of $\mathcal{L}(\mathcal{H}_D)$. This indicates that $\mathcal{L}(\mathcal{H}_D)$ plays the similar role in the category of Hilbert C^* -modules to that of $\mathbb{B}(\mathcal{H})$ in the category of C^* -algebras.

Proposition 2.5.9. *Let A be a separable C^* -algebra. Then there exists a faithful non-degenerate $*$ -homomorphism $\pi : A \rightarrow \mathcal{L}(\mathcal{H}_D)$.*

As we see, $\mathcal{L}(\mathcal{H}_D)$ plays the role of $\mathbb{B}(\mathcal{H})$. Note that $\mathbb{B}(\mathcal{H})$ is also a von Neumann algebra, but $\mathcal{L}(\mathcal{H}_D)$ is not in general. Paschke in [Pas73] introduces self-dual Hilbert C^* -modules to play the similar role in the von Neumann algebra context.

Let E be a Hilbert D -module. Each $x \in E$ gives rise to a bounded D -module map $\hat{x} : E \rightarrow D$ defined by $\hat{x}(y) = \langle x, y \rangle$ for $y \in E$. We will call E *self-dual* if every bounded D -module map of E into D arises by taking D -valued inner products with some $x \in E$. For instance, if D is unital, then it is a self-dual Hilbert D -module. Any self-dual Hilbert C^* -module is complete, but the converse is not true.

For von Neumann algebra N , it is natural to consider the self-dual Hilbert N -module E_N , because of the following theorem from [JS05].

Theorem 2.5.10. *For a Hilbert C^* -module E over a von Neumann algebra N , the following conditions are equivalent:*

1. *The unit ball of E is strongly closed;*
2. *E is principal, or equivalently, E is an ultraweak direct sum of Hilbert C^* -modules $q_\alpha N$, for some projections q_α ;*
3. *E is self-dual;*
4. *The unit ball of E is weakly closed.*

We denote the algebra of adjointable maps on E_N closed in the weak operator topology by $\mathcal{L}^w(E_N)$.

Remark 2.5.11. According to [Pas73] and Kasparov's absorption theorem, for any von Neumann algebra N , we have $\mathcal{L}^w(E_N) = e\mathbb{B}(\mathcal{H})\bar{\otimes}Ne$ for some projection e .

Remark 2.5.12. Let N be a von Neumann subalgebra of M , such that $N = zM$ for some central projection $z \in M$. Then one can unitize the inclusion map $\iota : \mathbb{B}(\ell_2)\bar{\otimes}N \hookrightarrow \mathbb{B}(\ell_2)\bar{\otimes}M$. Indeed since $\mathbb{B}(\ell_2)$ is a type I_∞ factor, the projection $1 \otimes z : \mathbb{B}(\ell_2)\bar{\otimes}M \rightarrow \mathbb{B}(\ell_2)\bar{\otimes}N$ is properly infinite, and hence it is equivalent to identity on $\mathbb{B}(\ell_2)\bar{\otimes}M$ [Tak1]. Let $1 \otimes z = v^*v$, and $\text{id}_{\mathbb{B}(\ell_2)\bar{\otimes}M} = vv^*$. Note that $(1 \otimes z) \circ \iota = \text{id}_{\mathbb{B}(\ell_2)\bar{\otimes}N}$. Multiplying by v from left and by v^* from right, we get $vvv^* = \text{id}_{\mathbb{B}(\ell_2)\bar{\otimes}N}$.

2.5.1 Kirchberg's observations on the multiplier algebras

In this section, we explore Kirchberg's seminal paper on non-semisplit extensions in detail. The proof of the following results will appear in Sepideh Rezvani's thesis.

Let A , B and C be C^* -algebras. We say a map $h : A \rightarrow B$ *factors through C approximately via u.c.p. maps in point-norm topology* if there exist u.c.p. maps $\varphi_n : A \rightarrow C$ and $\psi_n : C \rightarrow B$ such that the following diagram commutes approximately in point-norm topology:

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ & \searrow \varphi_n & \nearrow \psi_n \\ & C & \end{array}$$

i.e. $\|(\psi_n \circ \varphi_n)(x) - h(x)\| \rightarrow 0$ for all $x \in A$.

Theorem 2.5.13. *Let A be a C^* -algebra and $\mathcal{M}(A)$ be its multiplier algebra. Then the identity map on $\mathcal{M}(A)$ factors through $\ell_\infty(A)$ approximately via u.c.p. maps in point-norm topology. That is the following diagram commutes in point-norm topology.*

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\text{id}} & \mathcal{A} \\
 & \searrow \varphi_n & \nearrow \psi_n \\
 & \ell_\infty(A) &
 \end{array}$$

Using the above theorem, we can establish the following result on the relation between $\mathcal{M}(A)$ and A^{**} .

Corollary 2.5.14. *Suppose A is a C^* -algebra and $\mathcal{M}(A)$ is its multiplier algebra. Then $\mathcal{M}(A)$ is relatively weakly injective in A^{**} .*

Chapter 3

Operator-valued Kirchberg Theory

In [Ki93], Kirchberg shows that for two QWEP von Neumann algebras M and N , their free product $M * N$ is also QWEP. It is natural to investigate the property of being QWEP for an amalgamated free product of two QWEP algebras. It is known that for two QWEP von Neumann algebras M and N , and an amenable C^* -subalgebra A , the amalgamated free product $M *_A N$ is also QWEP. But it is not known for a general C^* -subalgebra A . To reduce the complexity of the problem, we consider the property of being QWEP relative to a C^* -algebra A . We are interested to see if in the case where both M and N are QWEP relative to A , whether $M *_A N$ is QWEP relative to A or not.

To study the notion of relative QWEP, first we need to define the relative WEP. Recall that in [Lan73], Lance defined a C^* -algebra A to have the WEP, if for $A \subset \mathbb{B}(\mathcal{H})$, A is relatively weakly injective (abbreviated as *r.w.i.*) in $\mathbb{B}(\mathcal{H})$, namely there exists a u.c.p. map from $\mathbb{B}(\mathcal{H})$ to A^{**} such that its restriction to A is the identity. To define the notion of the relative WEP, there are two natural ways of replacing $\mathbb{B}(\mathcal{H})$ in the framework of Hilbert C^* -module. Recall that for a C^* -algebra D , any C^* -algebra can be regarded as a C^* -subalgebra of $\mathcal{L}(\mathcal{H}_D)$, where \mathcal{H}_D is a Hilbert D -module, and $\mathcal{L}(\mathcal{H}_D)$ is the C^* -algebra of bounded adjointable D -linear maps on \mathcal{H}_D . Another way of representation is to replace $\mathcal{L}(\mathcal{H}_D)$ by the von Neumann algebra $\mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$. We say that A has the WEP_1 (WEP_2) relative to C^* -algebra D , if A is *r.w.i.* in $\mathcal{L}(\mathcal{H}_D)$ ($\mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$, respectively).

In this chapter, we investigate basic properties of these two notions. We discover that Kirchberg's methodology in his seminal work on non-semisplit extensions is functorial, and gives rise to properties as in the classical case. In particular, we establish the tensorial characterization for the two notions. Also we study the relation between the two notions of relative WEP. This leads to a more general question: Let A and B be C^* -algebras such that $A \subset B \subset A^{**}$ canonically. Does this imply that B is *r.w.i.* in A^{**} ? The answer to this question turns out to be negative in the general case. However, in the special case where $A = \mathbb{K} \otimes_{\min} D$ and $B = \mathcal{L}(\mathcal{H}_D)$, B is *r.w.i.* in $A^{**} = (\mathbb{K} \otimes_{\min} D)^{**} = \mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$. This shows that $D\text{WEP}_1$ implies $D\text{WEP}_2$. We also show that the converse is not true.

3.1 Module version of the weak expectation property

The notion of *r.w.i.* is a paired relation between a C^* -subalgebra and its parent C^* -algebra. If the parent C^* -algebra is $\mathbb{B}(\mathcal{H})$, the *r.w.i.* property is equivalent to the WEP. By carefully choosing a parent C^* -algebra, we can define the notion of WEP relative to a C^* -algebra.

Let \mathcal{C} be a collection of inclusions of unital C^* -algebras $\{(A \subseteq X)\}$.

For a C^* -algebra D , there are two classes of objects that we will discuss throughout this thesis.

1. $\mathcal{C}_1 = \{A \subseteq \mathcal{L}(E_D)\}$, where E_D is a Hilbert D -module.
2. $\mathcal{C}_2 = \{A \subseteq \mathcal{L}^w(E_{D^{**}})\}$, where $E_{D^{**}}$ is a self dual Hilbert D^{**} -module.

Definition 3.1.1. A C^* -algebra A is said to have the $DWEP_i$ for $i = 1, 2$, if there exists a pair of inclusions $A \subseteq X$ in \mathcal{C}_i such that A is relatively weakly injective in X .

Notice that the notion of the $DWEP$ is a *r.w.i.* property. By Corollary 2.2.6, the WEP implies the $DWEP_i$, for $i = 1, 2$. Also, inherited from *r.w.i.* property, we have the following lemmas for the $DWEP$.

Lemma 3.1.2. Let A_0 and A_1 be C^* -algebras such that A_0 is relatively weakly injective in A_1 . If A_1 has the $DWEP_i$ for $i = 1, 2$, then so does A_0 .

Proof. Since A_1 has the $DWEP_i$, there exists a pair of inclusions $A_1 \subseteq X$ in \mathcal{C}_i such that A_1 is *r.w.i.* in X , for $i = 1, 2$. By Lemma 2.2.3, A_0 is *r.w.i.* in X . Therefore the result follows. \square

Remark 3.1.3. By the absorption theorem and Remark 2.5.5 and 2.5.11, $\mathcal{L}(E_D)$ is *r.w.i.* in some $\mathcal{L}(\mathcal{H}_D)$ and $\mathcal{L}^w(E_{D^{**}})$ is *r.w.i.* in some $\mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$. Sometimes it is more convenient to consider the $DWEP_1$ as relatively weak injectivity in $\mathcal{L}(\mathcal{H}_D)$, and the $DWEP_2$ as relatively weak injectivity in $\mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$, because of the concrete structures.

Example 3.1.4. From the above, all WEP algebras have $DWEP_i$ for arbitrary C^* -algebra D . Also, D has the $DWEP_i$ trivially for 1-dimensional Hilbert space \mathcal{H} . Our first non-trivial example of $DWEP_i$ is $\mathbb{K} \otimes D$. For the first class \mathcal{C}_1 , $\mathbb{K} \otimes D$ is a principle ideal of $\mathcal{L}(\mathcal{H}_D)$, and thus is *r.w.i.* in $\mathcal{L}(\mathcal{H}_D)$. For the second class \mathcal{C}_2 , note that $(\mathbb{K} \otimes D)^{**} = \mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$, so $\mathbb{K} \otimes D$ is *r.w.i.* in $\mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$. By universality of $\mathcal{L}(\mathcal{H}_D)$ and $\mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$, $\mathbb{K} \otimes D$ has the $DWEP_i$ for both $i = 1, 2$.

Because of the injectivity of $\mathbb{B}(\mathcal{H})$, we see that the notion of the WEP does not depend on the representation $A \subseteq \mathbb{B}(\mathcal{H})$. By constructing a universal object in the classes \mathcal{C}_i , we can define the $DWEP_i$ independent of inclusions.

Lemma 3.1.5. *A C^* -algebra A has the $DWEP_i$ for some inclusion $A \subseteq X$ in \mathcal{C}_i , if and only if there exists a universal object X^u and $A \subseteq X^u$ in \mathcal{C}_i , such that*

1. *A is relatively weakly injective in X^u ;*
2. *If A is relatively weakly injective in some X , then there exists a u.c.p. map from X^u to X , which is identity on A .*

Proof. The “if” part is trivial. For the “only if” part, take \mathcal{C}_1 for example. The proof of the other case is similar. For all u.c.p. maps $\rho : A \rightarrow \mathcal{L}(E_D)$, by construction there exists a Hilbert D -module E_ρ and a $*$ -homomorphism $\pi_\rho : A \rightarrow \mathcal{L}(E_\rho)$. Let $E_D^u = \bigoplus_\rho E_\rho$. Then any $\mathcal{L}(E_D)$ containing A can be embedded into $\mathcal{L}(E_D^u)$, and there exists a truncation $\mathcal{L}(E_D^u) \rightarrow \mathcal{L}(E_D)$. Now suppose A is *r.w.i.* in some $\mathcal{L}(E_D)$. Then it is also *r.w.i.* in $\mathcal{L}(E_D^u)$. Hence we complete the proof. \square

Following Lance’s tensor product characterization Theorem 2.2.5, we have a similar result for the $DWEP_i$, for $i = 1, 2$. We only present the result for the first class. The other case can be proved similarly.

Let $A \subseteq \mathcal{L}(E_D^u)$ be the universal representation. We define a tensor norm \max_1^D on $A \otimes C^*\mathbb{F}_\infty$ to be the norm induced from the inclusion $A \otimes C^*\mathbb{F}_\infty \subseteq \mathcal{L}(E_D^u) \otimes_{\max} C^*\mathbb{F}_\infty$ isometrically. This induced norm is categorical in the sense that if ϕ is a u.c.p. map from A to B , then $\phi \otimes \text{id}$ extends a u.c.p. map from $A \otimes_{\max_1^D} C^*\mathbb{F}_\infty$ to $B \otimes_{\max_1^D} C^*\mathbb{F}_\infty$. Indeed, let ι be the inclusion map from B to its universal representation $\mathcal{L}^B(E_D^u)$, then $\iota \circ \phi$ is a u.c.p. map from A to $\mathcal{L}^B(E_D^u)$. By KSGNS and the construction of $\mathcal{L}^A(E_D^u)$, there exists a u.c.p. map from $\mathcal{L}^A(E_D^u)$ to $\mathcal{L}^B(E_D^u)$ extending the map $\iota \circ \phi$. Hence we have a composition of u.c.p. maps

$$A \otimes_{\max_1^D} C^*\mathbb{F}_\infty \subseteq \mathcal{L}^A(E_D^u) \otimes_{\max} C^*\mathbb{F}_\infty \rightarrow \mathcal{L}^B(E_D^u) \otimes_{\max} C^*\mathbb{F}_\infty,$$

whose image is $B \otimes_{\max_1^D} C^*\mathbb{F}_\infty$.

Theorem 3.1.6. *A C^* -algebra A has the $DWEP_1$, if and only if*

$$A \otimes_{\max_1^D} C^*\mathbb{F}_\infty = A \otimes_{\max} C^*\mathbb{F}_\infty.$$

Proof. First, suppose A has the $DWEP_1$, then A is *r.w.i.* in $\mathcal{L}(E_D^u)$. That is, there exists a u.c.p. map $\varphi : \mathcal{L}(E_D^u) \rightarrow A^{**}$ such that $\varphi|_A = \text{id}_A$. Then $\varphi \otimes \text{id}$ gives a u.c.p. map from $\mathcal{L}(E_D^u) \otimes_{\max} C^*\mathbb{F}_\infty$

to $A^{**} \otimes_{\max} C^*\mathbb{F}_\infty$. Therefore the map $\varphi|_A \otimes \text{id}$, defined on the algebraic tensor product, extends to a u.c.p. map from $A \otimes_{\max_1^D} C^*\mathbb{F}_\infty$ to $A^{**} \otimes_{\max} C^*\mathbb{F}_\infty$, whose image is $A \otimes_{\max} C^*\mathbb{F}_\infty$.

To prove the other direction, suppose $A \otimes_{\max_1^D} C^*\mathbb{F}_\infty = A \otimes_{\max} C^*\mathbb{F}_\infty$, and let $A \subseteq \mathbb{B}(\mathcal{H})$ be the universal representation, i.e. $A'' = A^{**}$. Let π be a representation of $C^*\mathbb{F}_\infty$ to $\mathbb{B}(\mathcal{H})$ with $\pi(C^*\mathbb{F}_\infty)'' = A'$. Then there exists a u.c.p. map $A \otimes_{\max_1^D} C^*\mathbb{F}_\infty = A \otimes_{\max} C^*\mathbb{F}_\infty \rightarrow \mathbb{B}(\mathcal{H})$. Since $A \otimes_{\max_1^D} C^*\mathbb{F}_\infty \subseteq \mathcal{L}(E_D^u) \otimes_{\max} C^*\mathbb{F}_\infty$, by Arveson's extension theorem, we can extend the above u.c.p. map to $\Psi : \mathcal{L}(E_D^u) \otimes_{\max} C^*\mathbb{F}_\infty \rightarrow \mathbb{B}(\mathcal{H})$. Now define a map ψ on $\mathcal{L}(E_D^u)$ by $\psi(T) := \Psi(T \otimes 1)$, for $T \in \mathcal{L}(E_D^u)$. Then ψ is a u.c.p. extension of id_A . Since Ψ is a $C^*\mathbb{F}_\infty$ -bimodule map, we have $\psi(T)\pi(x) = \Psi(T \otimes x) = \pi(x)\psi(T)$ for $T \in \mathcal{L}(E_D^u)$ and $x \in C^*\mathbb{F}_\infty$, i.e., $\psi(T) \in \pi(C^*\mathbb{F}_\infty)' = A^{**}$. This completes the proof. \square

It is natural to explore the relationship between the $DWEP_1$ and the $DWEP_2$. We have the following.

Theorem 3.1.7. *If a C^* -algebra A has the $DWEP_1$, then it also has the $DWEP_2$.*

In fact, the converse of the above theorem is not true, and we will give a counterexample in Section 5.

To prove the above theorem, we need following lemmas.

Lemma 3.1.8. *Suppose that the identity map on a C^* -algebra A factors through a C^* -algebra B approximately via u.c.p. maps in point-norm topology, i.e. there exist two nets of u.c.p. maps $\phi_i : A \rightarrow B$ and $\psi_i : B \rightarrow A$, such that $\|\psi_i \circ \phi_i(x) - x\| \rightarrow 0$ for $x \in A$. If B has the $DWEP_i$, then so does A .*

Proof. Following Kirchberg's method, it suffices to show $A \otimes_{\max_1^D} C^*\mathbb{F}_\infty = A \otimes_{\max} C^*\mathbb{F}_\infty$. Since we have u.c.p. maps $\phi_i : A \rightarrow B$ and $\psi_i : B \rightarrow A$, such that $\|\psi_i \circ \phi_i(x) - x\| \rightarrow 0$ for $x \in A$, we have u.c.p. maps $\phi_i \otimes \text{id} : A \otimes_{\max_1^D} C^*\mathbb{F}_\infty \rightarrow B \otimes_{\max_1^D} C^*\mathbb{F}_\infty$ and u.c.p. maps $\psi_i \otimes \text{id} : B \otimes_{\max} C^*\mathbb{F}_\infty \rightarrow A \otimes_{\max} C^*\mathbb{F}_\infty$. Since B has the $DWEP_i$, by Theorem 3.1.6, we have $B \otimes_{\max_1^D} C^*\mathbb{F}_\infty = B \otimes_{\max} C^*\mathbb{F}_\infty$. Therefore we have u.c.p. maps $A \otimes_{\max_1^D} C^*\mathbb{F}_\infty \rightarrow A \otimes_{\max} C^*\mathbb{F}_\infty$ defined by the composition of the maps according to the following diagram

$$A \otimes_{\max_1^D} C^*\mathbb{F}_\infty \xrightarrow{\phi_i \otimes \text{id}} B \otimes_{\max_1^D} C^*\mathbb{F}_\infty = B \otimes_{\max} C^*\mathbb{F}_\infty \xrightarrow{\psi_i \otimes \text{id}} A \otimes_{\max} C^*\mathbb{F}_\infty.$$

This net of maps converges to the identity. Hence we get the result. \square

Another lemma we need is that the $DWEP_i$ property is preserved under the direct product.

Lemma 3.1.9. *If $(A_i)_{i \in I}$ is a net of C^* -algebras with the $DWEP_i$, then $\prod_{i \in I} A_i$ has the $DWEP_i$.*

Proof. We will prove the result for the $DWEP_1$. The proof of the other case is similar. Since each A_i has the $DWEP_1$, there exists an inclusion $A_i \subseteq \mathcal{L}((E_i)_D)$ such that A_i is *r.w.i.* in $\mathcal{L}((E_i)_D)$. By Lemma 2.2.4, $\prod_{i \in I} A_i$ is *r.w.i.* in $\prod_{i \in I} \mathcal{L}((E_i)_D)$. Since $\mathcal{L}(\oplus_{i \in I} (E_i)_D)$ contains $\prod_{i \in I} \mathcal{L}((E_i)_D)$ and it has a conditional expectation onto $\prod_{i \in I} \mathcal{L}((E_i)_D)$, $\prod_{i \in I} A_i$ is also *r.w.i.* in $\mathcal{L}(\oplus_{i \in I} (E_i)_D)$. Therefore $\prod_{i \in I} A_i$ has the $DWEP_i$. \square

Kirchberg[Ki93] shows that for a C^* -algebra A , the multiplier algebra $\mathcal{M}(A)$ factors through $\ell_\infty(A)$ approximately by u.c.p. maps (Theorem 2.5.13). Using this fact, we have the following.

Corollary 3.1.10. *Suppose that the C^* -algebra A has the $DWEP_i$, for $i = 1, 2$. Then the multiplier algebra $\mathcal{M}(A)$ also has the $DWEP_i$, for $i = 1, 2$.*

Proof. By Theorem 2.5.13, $\mathcal{M}(A)$ factors through $\ell_\infty(A)$ approximately via u.c.p. maps in point-norm topology. Since A has the $DWEP_i$, $\ell_\infty(A)$ has $DWEP_i$ by Lemma 3.1.9. Therefore by Lemma 3.1.8, $\mathcal{M}(A)$ also has the $DWEP_i$. \square

Now we are ready to see the proof of the theorem.

Proof of Theorem 3.1.7. It suffices to show that $\mathcal{L}(\mathcal{H}_D)$ has the $DWEP_2$. Notice that $\mathcal{L}(\mathcal{H}_D) = \mathcal{M}(\mathbb{K} \otimes_{\min} D)$, and also $\mathcal{M}(\mathbb{K} \otimes_{\min} D)$ factors through $\ell_\infty(\mathbb{K} \otimes_{\min} D)$ approximately via u.c.p. maps in point-norm topology. Since $\mathbb{K} \otimes_{\min} D$ has the $DWEP_2$ by Remark 3.1.4, and hence $\ell_\infty(\mathbb{K} \otimes_{\min} D)$ by Lemma 3.1.9. By Corollary 3.1.10, $\mathcal{M}(\mathbb{K} \otimes_{\min} D)$ has the $DWEP_2$. \square

Remark 3.1.11. Note that $D^{**}WEP_1$ implies $DWEP_2$. Indeed having $D^{**}WEP_1$ is equivalent to being *r.w.i.* in $\mathcal{L}(\mathcal{H}_{D^{**}}) = \mathcal{M}(\mathbb{K} \otimes_{\min} D^{**})$, and having $DWEP_2$ is equivalent to being *r.w.i.* in $\mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$. Note that $\mathbb{K} \otimes_{\min} D^{**}$ is *r.w.i.* in $\mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$. By Corollary 3.1.10, we have $\mathcal{M}(\mathbb{K} \otimes_{\min} D^{**})$ has the $DWEP_2$ as well.

Now we investigate some properties of the module WEP. The first result is that the module WEP is stable under tensoring with a nuclear C^* -algebra, similar to the classical case.

Proposition 3.1.12. *For a C^* -algebra D , the following properties hold:*

1. *If a C^* -algebra A has the $DWEP_1$, and B is a nuclear C^* -algebra, then $A \otimes_{\min} B$ has the $DWEP_1$ as well.*

2. If von Neumann algebras M and N have the $CWEP_2$ and $DWEP_2$ respectively, then $M\bar{\otimes}N$ has the $(C \otimes_{\min} D)WEP_2$.

Proof. (1) Since A has the $DWEP_1$ and B is a nuclear, we have A is *r.w.i.* in $\mathcal{L}(\mathcal{H}_D)$, and B is *r.w.i.* in $\mathbb{B}(\mathcal{H})$. Therefore we have u.c.p. maps $A \otimes_{\min} B \rightarrow \mathcal{L}(\mathcal{H}_D) \otimes_{\min} \mathbb{B}(\mathcal{H}) = \mathcal{L}(\tilde{\mathcal{H}}_D) \rightarrow A^{**} \otimes_{\min} B^{**}$. Note that B is nuclear and hence exact, so the inclusion map $A^{**} \otimes B^{**} \hookrightarrow (A \otimes_{\min} B)^{**}$ is min-continuous. Therefore $A \otimes_{\min} B$ is *r.w.i.* in $\mathcal{L}(\tilde{\mathcal{H}}_D)$.

(2) Since M is *r.w.i.* in $\mathcal{L}^w(\mathcal{H}_{C^{**}})$ and N is *r.w.i.* in $\mathcal{L}^w(\mathcal{H}_{D^{**}})$, we have u.c.p. maps $M\bar{\otimes}N \rightarrow \mathcal{L}^w(\mathcal{H}_{C^{**}}) \bar{\otimes} \mathcal{L}^w(\mathcal{H}_{D^{**}}) = \mathcal{L}^w(\mathcal{H}_{C^{**} \bar{\otimes} D^{**}}) \rightarrow M\bar{\otimes}N \rightarrow (M\bar{\otimes}N)^{**}$. Note that $C \otimes_{\min} D$ is weak *-dense in $C^{**} \bar{\otimes} D^{**}$. Therefore we have a normal conditional expectation $(C \otimes_{\min} D)^{**} \rightarrow C^{**} \bar{\otimes} D^{**}$, and hence $C^{**} \bar{\otimes} D^{**}$ is *r.w.i.* in $(C \otimes_{\min} D)^{**}$. Therefore $\mathcal{L}^w(\mathcal{H}_{C^{**} \bar{\otimes} D^{**}})$ is *r.w.i.* in $\mathcal{L}^w(\mathcal{H}_{(C \otimes_{\min} D)^{**}})$, and hence $M\bar{\otimes}N$ is *r.w.i.* in $\mathcal{L}^w(\mathcal{H}_{(C \otimes_{\min} D)^{**}})$. \square

As a consequence of Corollary 3.1.10, we have the transitivity property of $DWEP$.

Proposition 3.1.13. *If A has the $BWEP_i$, and B has the $CWEP_i$, then A has the $CWEP_i$, for $i = 1, 2$.*

Proof. Since B has the $CWEP_i$, then so does $\mathbb{K} \otimes_{\min} B$, and hence so does $\mathcal{M}(\mathbb{K} \otimes_{\min} B)$ by Corollary 3.1.10. Since A has the $BWEP_i$, A is *r.w.i.* in some $\mathcal{L}(\mathcal{H}_B) = \mathcal{M}(\mathbb{K} \otimes_{\min} B)$. By the transitivity of *r.w.i.*, we conclude that A has the $CWEP_i$, for $i = 1, 2$. \square

Corollary 3.1.14. *If A has the $DWEP_1$, and D has the WEP , then A has the WEP .*

Proof. It suffices to show that $\mathcal{L}(\mathcal{H}_D) = \mathcal{M}(\mathbb{K} \otimes_{\min} D)$ has the WEP . This is obvious since if D has the WEP , then so does $\mathbb{K} \otimes_{\min} D$ and hence $\mathcal{M}(\mathbb{K} \otimes_{\min} D)$ has the WEP . \square

Remark 3.1.15. The previous result is not necessarily true for the WEP_2 case, since $\mathbb{B}(\ell_2) \bar{\otimes} D^{**}$ may not have the WEP , for instance for $D = \mathbb{B}(\ell_2)$. See Example 3.3.1 for the proof.

In [Ju96], Junge shows the following finite dimensional characterization of the WEP .

Theorem 3.1.16. *The C^* -algebra A has the WEP if and only if for arbitrary finite dimensional subspaces $F \subset A$ and $G \subset A^*$, and $\varepsilon > 0$, there exist matrix algebra M_m and u.c.p. maps $u : F \rightarrow M_m$, $v : M_m \rightarrow A/G^\perp$, such that*

$$\|v \circ u - q_G \circ \iota_F\| < \varepsilon,$$

where $\iota_F : F \rightarrow A$ is the inclusion map and $q_G : A \rightarrow A/G^\perp$ is the quotient map.

We have a similar result for the module WEP as follows.

Theorem 3.1.17. *The C^* -algebra A has the $DWEP_1$ if and only if for arbitrary finite dimensional subspaces $F \subset A$ and $G \subset A^*$, and $\varepsilon > 0$, there exist matrix algebra $M_m(D)$ and u.c.p. maps $u : A \rightarrow M_m(D)$, $v : M_m(D) \rightarrow A/G^\perp$, such that*

$$\|v \circ u|_F - q_G \circ \iota_F\| < \varepsilon,$$

where $\iota_F : F \rightarrow A$ is the inclusion map and $q_G : A \rightarrow A/G^\perp$ is the quotient map.

For the $DWEP_2$ case, we will replace the matrix algebra $M_m(D)$ by $M_m(D^{**})$.

Proof. \Leftarrow : From the assumption, we get a net of maps u and v over (F, G, ε) . Taking the direct product of all such u , and one w^* -limit of v , we have u.c.p. maps $A \rightarrow \Pi_{(F, G, \varepsilon)} M_m(D) \rightarrow A^{**}$, whose composition is identity on A , and hence A is *r.w.i.* in $\Pi_{(F, G, \varepsilon)} M_m(D)$. By Lemma 3.1.9, $\Pi_{(F, G, \varepsilon)} M_m(D)$ has the $DWEP_1$ since $M_m(D)$ does. Therefore A has the $DWEP_1$.

\Rightarrow : A has the $DWEP_1$, and hence we have $A \rightarrow \mathcal{L}(\mathcal{H}_D) \rightarrow A^{**}$. Let σ_I be the composition of the inclusion maps

$$A \hookrightarrow \mathcal{L}(\mathcal{H}_D) \hookrightarrow \Pi_I(M_{m(i)}(D)) \hookrightarrow \Pi_I(M_{m(i)}(D^{**})).$$

Note that each of the inclusions above is *r.w.i.*. By taking the duals, we have

$$\varphi_I : A^* \xrightarrow{r.w.i.} \mathcal{L}(\mathcal{H}_D)^* \xrightarrow{r.w.i.} \Pi_I(M_{m(i)}(D))^* \xrightarrow{r.w.i.} \Pi_I(M_{m(i)}(D^{**}))^* = \ell_1^I(\mathcal{S}_1^{m(i)}(D^*))^{**}.$$

By the local reflexivity principle, for arbitrary F , G and ε as in the theorem, there exists a map $\alpha_I^\varepsilon : G \rightarrow \ell_1(I, \mathcal{S}_1(D^*))$, such that

$$|\langle \alpha_I^\varepsilon(g), \sigma_I(f) \rangle - \langle \varphi_I(g), \sigma_I(f) \rangle| < \varepsilon \|f\| \|g\|,$$

for $f \in F$ and $g \in G$. By carefully choosing an Auerbach basis for the finite dimensional spaces, we can have the above relation on a finite subset $I_0 \subset I$, i.e.

$$|\langle \alpha_{I_0}^\varepsilon(g), \sigma_{I_0}(f) \rangle - \langle \varphi_{I_0}(g), \sigma_{I_0}(f) \rangle| < \varepsilon \|f\| \|g\|.$$

By the *r.w.i.* property of σ_I we have $\langle \varphi_I(g), \sigma_I(f) \rangle = \langle g, f \rangle$. Therefore for f and g with norm 1, we

have $|\langle \alpha_{I_0}^\varepsilon(g), \sigma_{I_0}(f) \rangle - \langle g, f \rangle| < \varepsilon$, and hence $|\langle g, \alpha_{I_0}^{\varepsilon*} \circ \sigma_{I_0}(f) - f \rangle| < \varepsilon$. Let $u = \sigma_{I_0}$ and $v = \alpha_{I_0}^{\varepsilon*}$. Then we have the desired result. \square

3.2 Module version of QWEP

In this section, we will follow the two notions of module WEP to define the module QWEP. This is joint work with Sepideh Rezvani. Since proofs of these results will appear in her thesis, we will list the results without proofs.

Definition 3.2.1. *A C^* -algebra B is said to be $DQWEP_i$ if it is the quotient of a C^* -algebra A with $DWEP_i$ for $i = 1, 2$.*

Similar to the $DWEP_i$, we have a tensor characterization for $DQWEP_i$ for $i = 1, 2$ as follows. First we need the following result due to Kirchberg.

Lemma 3.2.2 ([Ki93] Corollary 3.2 (v)). *If $\phi : A \rightarrow B^{**}$ is a u.c.p. map such that ϕ maps the multiplicative domain $\text{md}(\phi)$ of ϕ onto a C^* -subalgebra C of B^{**} containing B as a subalgebra, then the C^* -algebra $\text{md}(\phi) \cap \phi^{-1}(B)$ is relatively weakly injective in A .*

We only prove the tensor characterization for $DQWEP_1$. The proof of the other case is similar.

Theorem 3.2.3. *Let $C^*\mathbb{F}_\infty \subset \mathcal{L}(\mathcal{H}_D^u)$ be the universal representation. The following statements are equivalent:*

- (i) *A C^* -algebra B is $DQWEP_1$;*
- (ii) *For any u.c.p. map $u : C^*\mathbb{F}_\infty \rightarrow B$, the map $u \otimes \text{id}$ extends to a continuous map from $C^*\mathbb{F}_\infty \otimes_{\max_1^D} C^*\mathbb{F}_\infty$ to $B \otimes_{\max} C^*\mathbb{F}_\infty$, where \max_1^D is the induced norm from the inclusion $C^*\mathbb{F}_\infty \otimes C^*\mathbb{F}_\infty \subseteq \mathcal{L}(\mathcal{H}_D^u) \otimes_{\max} C^*\mathbb{F}_\infty$.*

Remark 3.2.4. In the proof of the above Theorem, we showed that the second statement is equivalent to the statement that for any u.c.p. map $u : C^*\mathbb{F}_\infty \rightarrow B$, $w : C^*\mathbb{F}_\infty \rightarrow B^{\text{op}}$, the map $u \otimes w$ extends to a continuous map from $C^*\mathbb{F}_\infty \otimes_{\max_1^D} C^*\mathbb{F}_\infty$ to $B \otimes_{\max} B^{\text{op}}$.

Now let us investigate some basic properties of the $DQWEP$. We have the following proposition, similar to the $DWEP$ case.

Proposition 3.2.5. *The following hold:*

1. If a C^* -algebra B is $DQWEP_1$ and C is nuclear, then $C \otimes_{\min} B$ is also $DQWEP_1$.
2. If von Neumann algebras M and N are $CQWEP_2$ and $DQWEP_2$, respectively, then $M \bar{\otimes} N$ is $(C \otimes_{\min} D)QWEP_2$.

By Theorem 3.1.7, $DWEP_1$ implies $DWEP_2$, and hence $DQWEP_1$ implies $DQWEP_2$. In Section 5 we will show that there exist C^* -algebras with $DWEP_2$ which do not have $DWEP_1$. However in the $QWEP$ context, the two concepts coincide. To see this, we need the following lemmas in which we use Kirchberg's categorical method.

Remark 3.2.6. If a C^* -algebra A has the $DWEP_2$, then it is $D^{**}QWEP_1$. Indeed since A has the $DWEP_2$, it is *r.w.i.* in $\mathbb{B}(\ell_2) \bar{\otimes} D^{**} = (\mathbb{K} \otimes_{\min} D)^{**}$. Now since D is $D^{**}QWEP_1$, so is $\mathbb{K} \otimes_{\min} D$ and therefore, so is $(\mathbb{K} \otimes_{\min} D)^{**}$. Hence A is $D^{**}QWEP_1$.

The next lemma shows that $DQWEP_i$, for $i = 1, 2$, is stable under the direct products.

Lemma 3.2.7. Suppose $(B_i)_{i \in I}$ is a net of C^* -algebras in $\mathbb{B}(\mathcal{H})$. If B_i is $DQWEP_i$, for all $i \in I$, then so is $\Pi_{i \in I} B_i$.

Lemma 3.2.8. Let B be a $DQWEP_i$ C^* -algebra, for $i = 1, 2$, and B_0 a C^* -subalgebra of B which is relatively weakly injective in B . Then B_0 is also a $DQWEP_i$ C^* -algebra.

Lemma 3.2.9. Let A and B be unital C^* -algebras. Suppose there exists a map $\psi : A \rightarrow B$ which maps the closed unit ball of A onto the closed unit ball of B . If A has the $DWEP_i$, then B is $DQWEP_i$, for $i = 1, 2$.

Corollary 3.2.10. Let B and C be C^* -algebras. Suppose B is $DQWEP_i$, and $\psi : B \rightarrow C$ is a u.c.p. map that maps the closed unit ball of B onto that of C . Then C is $DQWEP_i$.

Lemma 3.2.11. Suppose $(B_i)_{i \in I}$ is an increasing net of C^* -algebras in $\mathbb{B}(\mathcal{H})$. If all B_i are $DQWEP_i$, then $\overline{\cup B_i}$ and $(\cup B_i)''$ are $DQWEP_i$.

The next corollary shows that unlike the $DWEP$ case, the $DQWEP$ of a C^* -algebra and its double dual are equivalent.

Corollary 3.2.12. A C^* -algebra B is $DQWEP_i$ if and only if B^{**} is $DQWEP_i$ for $i = 1, 2$.

Lemma 3.2.13. Suppose B and C are C^* -algebras, and B factors through C approximately via u.c.p. maps in the point-weak* topology. If C is $DQWEP_i$, then so is B .

Corollary 3.2.14. *If a C^* -algebra B is $DQWEP_i$, for $i = 1, 2$, then so is $\mathcal{M}(B)$.*

We have the following transitivity result for $DQWEP_i$. We only show the $DQWEP_1$ case. The proof of the other case is similar. First we need the following lemma.

Lemma 3.2.15. *Let D be a C^* -algebra. If D is $CQWEP_i$ for $i = 1, 2$, then so are $\mathcal{L}(\mathcal{H}_D)$ and $\mathcal{L}^w(\mathcal{H}_{D^{**}})$.*

The following result shows the transitivity of the $DQWEP_i$ for $i = 1, 2$.

Corollary 3.2.16. *Let B , C and D be C^* -algebras such that B is $DQWEP_i$, and D is $CQWEP_i$. Then B is $CQWEP_i$.*

Now we are ready to establish the equivalence between the $DQWEP$ notions by observing the following result.

Theorem 3.2.17. *For a C^* -algebra B , the following conditions are equivalent:*

1. B is $DQWEP_1$;
2. B is $DQWEP_2$;
3. B^{**} is $D^{**}QWEP_1$;
4. B^{**} is $D^{**}QWEP_2$.

Proof. (1) \Rightarrow (2): This follows from the fact that $DWEP_1$ implies $DWEP_2$.

(2) \Rightarrow (3): Suppose B is $DQWEP_2$. Therefore, B is the quotient of a C^* -algebra A which is *r.w.i.* in $\mathcal{L}^w(E_{D^{**}})$. By Remark 3.2.6, since $\mathcal{L}^w(E_{D^{**}})$ has the $D^{**}WEP_1$, it is $D^{**}QWEP_1$. Hence A is $D^{**}QWEP_1$, and therefore, B is $D^{**}QWEP_1$.

(3) \Rightarrow (4): Follows from (1) \Rightarrow (2).

(4) \Rightarrow (1): Suppose B^{**} is $D^{**}QWEP_2$, and therefore so is B by Corollary 3.2.12. Then B is the quotient of a C^* -algebra A which is *r.w.i.* in $\mathcal{L}^w(E_{D^{****}})$. We have

$$A \stackrel{r.w.i.}{\subset} \mathcal{L}^w(E_{D^{****}}) \stackrel{r.w.i.}{\subset} \mathbb{B}(\ell_2) \bar{\otimes} D^{****} = (\mathbb{K} \otimes_{\min} D^{**})^{**}.$$

Therefore, it suffices to show that $\mathbb{K} \otimes_{\min} D^{**}$ is $DQWEP_1$. Notice that $\mathbb{K} \otimes_{\min} D^{**}$ factors through $\prod_n M_n(D^{**})$ approximately via u.c.p. maps in point-norm topology, since $\cup M_n(D^{**})$ is norm-dense in $\mathbb{K} \otimes_{\min} D^{**}$. Now since D has the $DWEP_1$, D^{**} is $DQWEP_1$. Therefore, by Proposition 3.2.5, so

is $M_n(D^{**}) = M_n \otimes_{\min} D^{**}$. Hence by Lemma 3.2.13, $\mathbb{K} \otimes_{\min} D^{**}$ is $DQWEP_1$. This finishes the proof. \square

3.3 Illustrations

In Section 3, we showed that $DWEP_1$ implies $DWEP_2$. Our first example will show the converse is not true, and hence the two notions of $DWEP$ are not equivalent.

Example 3.3.1. Let $D = \mathbb{B}(\ell_2)$. Note that $\mathcal{L}(\mathcal{H}_D) = \mathcal{M}(\mathbb{K} \otimes_{\min} \mathbb{B}(\ell_2))$, and $\mathbb{K} \otimes_{\min} \mathbb{B}(\ell_2)$ has the WEP, and so does $\mathcal{M}(\mathbb{K} \otimes_{\min} \mathbb{B}(\ell_2))$. Therefore the two notions of $DWEP_1$ and WEP coincide. On the other hand, the $DWEP_2$ of a C^* -algebra is the same as being *r.w.i.* in $\mathbb{B}(\mathcal{H}) \bar{\otimes} \mathbb{B}(\ell_2)^{**}$. Notice that $\mathbb{B}(\mathcal{H}) \bar{\otimes} \mathbb{B}(\ell_2)^{**} = (\mathbb{K} \otimes \mathbb{B}(\ell_2))^{**}$ is QWEP. Therefore by Proposition 2.2.9, $DWEP_2$ is equivalent to QWEP. Hence if A is a QWEP C^* -algebra without the WEP, for instance $C_\lambda^* \mathbb{F}_n$, then A has the $DWEP_2$ but not the $DWEP_1$, for $D = \mathbb{B}(\ell_2)$.

Now we are ready to see some examples of relative WEP and QWEP over special classes of C^* -algebras.

Proposition 3.3.2. *Let D be a nuclear C^* -algebra. Then a C^* -algebra A has the $DWEP_i$ for $i = 1, 2$ if and only if it has the WEP.*

Proof. Suppose A has the WEP. Therefore A has the $DWEP_1$, and hence the $DWEP_2$.

Now assume A has the $DWEP_2$, i.e. it is *r.w.i.* in $\mathbb{B}(\ell_2) \bar{\otimes} D^{**}$. Since D is nuclear, D^{**} is injective. Hence we have $D^{**} \subseteq \mathbb{B}(\mathcal{H}) \xrightarrow{\mathbb{E}} D^{**}$, where \mathbb{E} is a conditional expectation. Let $CB(A, B)$ be the space of completely bounded maps from A to B . Therefore we have

$$CB(S_1, D^{**}) \xrightarrow{\pi} CB(S_1, \mathbb{B}(\mathcal{H})) \xrightarrow{\varphi} CB(S_1, D^{**}),$$

where S_1 is the algebra of trace class operators, π is a $*$ -homomorphism, and φ acts by composing the maps in $CB(S_1, \mathbb{B}(\mathcal{H}))$ and \mathbb{E} . Note that by operator space theory $CB(S_1, D^{**}) \simeq \mathbb{B}(\ell_2) \bar{\otimes} D^{**}$ and $CB(S_1, \mathbb{B}(\mathcal{H})) \simeq \mathbb{B}(\ell_2) \bar{\otimes} \mathbb{B}(\mathcal{H}) = \mathbb{B}(\ell_2 \otimes \mathcal{H})$. Hence we have the maps $\mathbb{B}(\ell_2) \bar{\otimes} D^{**} \xrightarrow{\pi} \mathbb{B}(\ell_2) \bar{\otimes} \mathbb{B}(\mathcal{H}) = \mathbb{B}(\ell_2 \otimes \mathcal{H}) \xrightarrow{\varphi} \mathbb{B}(\ell_2) \bar{\otimes} D^{**}$. Now by Remark 2.5.12 we can unitize these two maps. Therefore A is *r.w.i.* in $\mathbb{B}(\ell_2 \otimes \mathcal{H})$, and hence it has the WEP. \square

After nuclear C^* -algebras, it is natural to consider the relative WEP for an exact C^* -algebra D . For convenience, we consider the following stronger version of the weak exactness property. A von

Neumann algebra $M \subseteq \mathbb{B}(\mathcal{H})$ is said to be *algebraically weakly exact*, (a.w.e. for short), if there exists a weakly dense exact C^* -algebra D in M . By [Ki94], we know that the a.w.e. implies the weak exactness.

Notice that the unitization trick works better in \mathcal{C}_2 category, and hence we have the following.

Proposition 3.3.3. *A C^* -algebra has the $DWEP_2$ for some exact C^* -algebra D if and only if it is relatively weakly injective in an a.w.e. von Neumann algebra.*

Proof. Suppose a C^* -algebra A has the $DWEP_2$, then A is *r.w.i.* in $\mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$. Since both \mathbb{K} and D are exact C^* -algebras, so is $\mathbb{K} \otimes_{\min} D$. Note that $\mathbb{K} \otimes_{\min} D$ is weakly dense in $(\mathbb{K} \otimes_{\min} D)^{**} = \mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$. We have $\mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$ is a.w.e.

For the other direction, suppose A is *r.w.i.* in an a.w.e. von Neumann algebra M . Let D be an exact C^* -algebra with $D'' = M$. Then there exists a central projection z in D^{**} such that $M = zD^{**}$. Hence we have completely positive maps $M \hookrightarrow D^{**} \rightarrow M$, which preserves the identity on M . Therefore by unitization M is *r.w.i.* in $\mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$ for some infinite dimensional Hilbert space \mathcal{H} . Hence if A is *r.w.i.* in M , then it is also *r.w.i.* in $\mathbb{B}(\mathcal{H}) \bar{\otimes} D^{**}$, and therefore it has the $DWEP_2$. \square

As we see, the nuclear-WEP is equivalent to the WEP. But the exact-WEP is different.

Example 3.3.4. Let \mathbb{F}_2 be the free group of two generators. Then it is exact and hence $C_\lambda^* \mathbb{F}_2$ is exact and $L\mathbb{F}_2$ is weakly exact. Since $C_\lambda^* \mathbb{F}_2$ is *r.w.i.* in $L\mathbb{F}_2$, by Proposition 3.3.3, $C_\lambda^* \mathbb{F}_2$ has the $DWEP_2$ for $D = C_\lambda^* \mathbb{F}_2$. But $C_\lambda^* \mathbb{F}_2$ does not have the WEP, since the WEP of a reduced group C^* -algebra is equivalent to the amenability of the group (see Proposition 3.6.9 in [BrOz]).

Now we consider the full group C^* -algebra of free group $C^* \mathbb{F}_\infty$. Since it is universal in the sense that for any unital separable C^* -algebra A , we have a quotient map $q : C^* \mathbb{F}_\infty \rightarrow A$. By the unitization trick, we have the following.

Proposition 3.3.5. *Let A be a unital separable C^* -algebra. Then it has the $DWEP_2$ for $D = C^* \mathbb{F}_\infty$.*

Proof. Since we have a quotient map $q : C^* \mathbb{F}_\infty \rightarrow A$, there exists a central projection z in $C^* \mathbb{F}_\infty^{**}$ such that $A^{**} = zC^* \mathbb{F}_\infty^{**}$. Hence we have an embedding $A^{**} \hookrightarrow \mathbb{B}(\mathcal{H}) \bar{\otimes} C^* \mathbb{F}_\infty^{**}$ with a completely positive map from $\mathbb{B}(\mathcal{H}) \bar{\otimes} C^* \mathbb{F}_\infty^{**}$ to A^{**} by multiplying $1 \otimes z$. By the unitization trick in Remark 2.5.12, A^{**} has the $DWEP_2$ for $D = C^* \mathbb{F}_\infty$ and so does A , since A is *r.w.i.* in A^{**} . \square

It is natural and even more interesting to ask whether the full group C^* -algebra $C^* \mathbb{F}_\infty$ has $DWEP$, for D is the reduced group C^* -algebra $C_\lambda^* \mathbb{F}_2$. In fact, this is related to the QWEP conjecture. If $C^* \mathbb{F}_\infty$

has the $DWEP_1$ for some WEP algebra D , then it has the WEP by Corollary 3.1.14 of transitivity. If $C^*\mathbb{F}_\infty$ does not have the $DWEP_1$ for some C^* -algebra D , then it does not have the WEP either. At the time of writing, we do not have an answer for this question.

Now let us discuss some properties of being module QWEP relative to some special classes of C^* -algebras. In the rest of this section, we will examine the relation between one of the equivalent statements of Theorem 3.2.17 (for example statement (1), B is $DQWEP_i$), and the statement that B^{**} is $D^{**}WEP_i$, for either $i = 1$ or 2 .

Proposition 3.3.6. *Let B be a C^* -algebra. If B^{**} has the $D^{**}WEP_i$, then B is $DQWEP_i$, for $i = 1, 2$.*

Proof. Suppose B^{**} has the $D^{**}WEP_i$, and hence B^{**} is $D^{**}QWEP_i$ by the trivial quotient. By Theorem 3.2.17, B is $DQWEP_i$. \square

For some C^* -algebra D , the four equivalent statements in Theorem 3.2.17 are equivalent to the statement that B^{**} has the $D^{**}WEP_i$. But this is not true in general. We will show examples of both circumstances.

Example 3.3.7. Let $D = \mathbb{B}(\ell_2)$. Then a C^* -algebra B is $DQWEP_i$ if and only if B^{**} has the $D^{**}WEP_i$, since they are both equivalent to B being QWEP. Indeed, if B is $DQWEP_1$, then $B = A/J$ and A has the $DWEP_1$. Since $\mathcal{L}(\mathcal{H}_D)$ has the WEP as shown in Example 3.3.1, so does A , and hence B is QWEP. On the other hand, having $\mathbb{B}(\ell_2)^{**}WEP_1$ is equivalent to being *r.w.i.* in $\mathcal{M}(\mathbb{K} \otimes_{\min} \mathbb{B}(\ell_2)^{**})$, which is QWEP. Hence B^{**} is QWEP. By Proposition 2.2.8, B is QWEP as well.

Example 3.3.8. Let D be a nuclear C^* -algebra. Then the above statements are not equivalent. Indeed, it follows from Proposition 3.3.2 that a C^* -algebra is $DQWEP_i$ if and only if it is QWEP. On the other hand, assume that B^{**} has the $D^{**}WEP_1$. Note that $D^{**}WEP_1$ implies $DWEP_2$ by Remark 3.1.11, which is equivalent to WEP by Proposition 3.3.2, and B^{**} has the WEP if and only if it is injective. Therefore the fact that a C^* -algebra B is $DQWEP_i$ does not imply that B^{**} has the $D^{**}WEP_1$.

Example 3.3.9. For a von Neumann algebra M , let us compare the properties $MQWEP_1$ of B and the $M^{**}WEP_1$ of B^{**} . We have the following partial results.

Case (i): M is of type I_n . Then M is subhomogeneous, which is equivalent to nuclearity. By Example 3.3.8, these two statements are not equivalent.

Case (ii): M is of type I_∞ , then $\mathbb{B}(\ell_2) \bar{\otimes} M$ is *r.w.i.* in M . Suppose B is $MQWEP_1$, then B is a quotient of a C^* -algebra A which is *r.w.i.* in $\mathbb{B}(\ell_2) \bar{\otimes} M$. Hence B^{**} is *r.w.i.* in A^{**} and hence in $(\mathbb{B}(\ell_2) \bar{\otimes} M)^{**}$, and hence in M^{**} . Since M^{**} is isomorphic to $\mathcal{L}(\mathcal{H}_{M^{**}})$ for 1-dimensional Hilbert space \mathcal{H} , it follows that B^{**} has the $M^{**}WEP_1$.

Case (iii): M is of type II_∞ or III , then $\mathbb{B}(\ell_2) \bar{\otimes} M \simeq M$. By a similar argument to that of Case (ii), we have the same conclusion.

Case (iv): M is of type II_1 and a McDuff factor, i.e. $M \bar{\otimes} R \simeq M$. Then we have a completely positive map from $M \bar{\otimes} \mathbb{B}(\ell_2)$ to M by the following:

$$M \bar{\otimes} \mathbb{B}(\ell_2) \rightarrow M \bar{\otimes} \prod_{n=1}^{\infty} M_n \rightarrow M \bar{\otimes} R \bar{\otimes} L_\infty[0, 1] \subseteq M \bar{\otimes} R \bar{\otimes} R \simeq M \bar{\otimes} R \simeq M.$$

with a completely positive left inverse from M to $M \bar{\otimes} \mathbb{B}(\ell_2)$, namely $M \bar{\otimes} \mathbb{B}(\ell_2)$ factors through M by completely positive maps. Therefore $M \bar{\otimes} \mathbb{B}(\ell_2)$ is *r.w.i.* in M^{**} . By the same argument above, the equivalence is established.

At the time of writing, we do not have an affirmative answer for the case where M is a non-McDuff II_1 factor.

Chapter 4

Correspondence

A celebrated theorem of Connes [Con76] characterizes hyperfinite von Neumann algebras.

Theorem 4.0.10 (Connes). *Let $N \subset \mathbb{B}(\mathcal{H})$ be a von Neumann algebra. The following are equivalent:*

1. *There exists a norm one projection $E : \mathbb{B}(\mathcal{H}) \rightarrow N$.*
2. *N is hyperfinite.*
3. *There exist a net of normal states $\omega_n : \mathbb{B}(\mathcal{H}) \rightarrow \mathbb{C}$, such that:*

(a) *for arbitrary $u \in \mathcal{U}(\mathcal{H})$, $\|\omega_n u - u \omega_n\|_{\mathcal{S}_1(\mathcal{H})} \rightarrow 0$;*

(b) *$\omega_n(x) \rightarrow \tau(x)$ for $x \in N$.*

If we replace $\mathbb{B}(\mathcal{H})$ by M , for a pair of von Neumann algebras $N \subset M$, one finds many situations where there exists a norm one projection E from M onto N , and one may ask what properties of M are automatically inherited by N . This question is related to the concept of correspondence between two von Neumann algebras, which has been introduced by Connes and Jones in [Con], [CJ85], as a very useful tool for the study of type II_1 factors. Later Popa has systematically developed this point of view to get some new insights in the area [Pop]. Among many interesting results and remarks, he discussed Connes' classical work on the injective II_1 factor in the framework of correspondences, and he defined and studied a natural notion of amenability for a finite von Neumann algebra M relative to a von Neumann subalgebra N .

The most well-known inherited property from the norm one projection $E : M \rightarrow N$ is the amenability of M . In [AD95] Anantharaman-Delaroche shows that other approximation properties such as the weak*-completely bounded approximation property ([Haa86], [CH]), or the σ -weak approximation property of [Kra91] are also preserved. Their common feature is the approximation of the identity map of M by appropriate σ -weakly continuous bounded maps, and the main problem is that the norm one projection E is not σ -weakly continuous in general.

The existence of E follows easily from the existence of a net $(\phi_i)_{i \in I}$ of σ -weakly continuous completely positive contractions $\phi_i : M \rightarrow N$ such that $\lim_i \phi_i(x) = x$ σ -weakly for all $x \in N$. As is indicated in [AD95], the converse is very likely true, but Anantharaman-Delaroche does not prove it in full generality. Instead she shows that, when there exists a norm one projection E from M onto N , we may find a von Neumann algebra M_1 , Morita equivalent to M , which contains N as a von Neumann subalgebra, in such a way that there exists a net $(\phi_i)_{i \in I}$ of σ -weakly continuous completely positive contractions $\phi_i : M_1 \rightarrow N$ such that $\lim_i \phi_i(x) = x$ σ -weakly for all $x \in N$.

In this chapter, we will show that these two statements are equivalent, with the bridge of amenable correspondence as in [AD95]. The main theorem is the following:

Theorem 4.0.11. *Let $N \subset M$ be von Neumann algebras, then the following are equivalent:*

1. *The Hilbert $N - N$ correspondence ${}_N L_2(N)_N$ is weakly contained in ${}_N L_2(M)_N$.*
2. *There exist normal u.c.p. maps $\phi_i : M \rightarrow N$, such that $\phi_i(x) \rightarrow x$ in point- $\sigma(N, N_*)$ topology.*
3. *There exists a norm one projection $E : M \rightarrow N$.*

Anantharaman-Delaroche in [AD95] shows that $(1) \Rightarrow (2) \Rightarrow (3)$. We will show that $(3) \Rightarrow (1)$ to complete the equivalences.

In Section 4, we will apply the theorem to relative amenability. The notion of relative amenability was introduced in [OP10], where they characterize the equivalent conditions as in Connes' classical result on amenability. Also, Popa and Vaes in [PV14] show the equivalent relation between relative amenability and left amenability for certain correspondences in the sense of [AD95]. Their main concerns are in the case of finite von Neumann algebras. We will extend the equivalent theorem in the context of general von Neumann algebras.

In the last section, we will examine the relation between relative weak injectivity, short as *r.w.i.*, raised by Kirchberg in [Ki93], and weak containment of correspondences. Recall the notion of relatively weak injectivity from Kirchberg [Ki93]. For C^* -subalgebra $A \subseteq B$, we say that A is *relatively weakly injective* in B , if there exists a u.c.p. map $\varphi : B \rightarrow A^{**}$, such that $\varphi|_A = \text{id}_A$. The main problem of this section is the following: suppose that a C^* -algebra A represented faithfully in a von Neumann algebra M with N being the weak closure of A in M , if the inclusion $A \subset M$ is *r.w.i.*, can we find a conditional expectation from M to N ? This problem is related to the statement whether the Hilbert $A - N$ correspondence ${}_A L_2(N)_N$ is weakly contained in ${}_A L_2(M)_N$. It turns out the problem has a

negative answer. In particular, we find an example that $A \subset B \subset M$, in which A and B are C^* -algebras sitting in a von Neumann algebra M , such that $A \subset M$ is *r.w.i.*, but $B \subset M$ is not.

4.1 Preliminaries

4.1.1 Correspondences

For the reader's convenience, we first recall the basic facts on correspondences that will be needed in this chapter. For more details, the reader may consult [CJ85], [BDH], [Rie74], [Pop], [Pas73].

Let M and N be two von Neumann algebras. A *correspondence* from M to N is a Hilbert space \mathcal{H} with a pair of commuting normal representations π_M and $\pi_{N^{\text{op}}}$ of M and N^{op} (the opposite of N) respectively [CJ85]. Usually, the triple $(\mathcal{H}, \pi_M, \pi_{N^{\text{op}}})$ will be denoted by \mathcal{H} . For $x \in M$, $y \in N$ and $h \in \mathcal{H}$, we shall write xhy instead of $\pi_M(x)\pi_{N^{\text{op}}}(y)h$. In case of ambiguity on which algebras are acting, we shall write ${}_M\mathcal{H}_N$ instead of \mathcal{H} . The commutants of $\pi_M(M)$ and $\pi_{N^{\text{op}}}(N^{\text{op}})$ respectively will be denoted by $\mathcal{L}_M(\mathcal{H})$ and $\mathcal{L}_{N^{\text{op}}}(\mathcal{H})$. In this chapter, we shall always assume that π_M and $\pi_{N^{\text{op}}}$ are faithful.

The standard form [Haa85] of M gives rise to a correspondence $L_2(M)$ from M to M , called the identity correspondence. We will denote by J_M the conjugate linear isometry of $L_2(M)$ given with the standard form of M . In this example, we have $\pi_{M^{\text{op}}}(x) = J_M \hat{x} J_M$.

Let us recall another useful equivalent way to look at correspondences. Let X be a self-dual (right) Hilbert N -module (see [Pas73]). The N -valued inner product, denoted by $\langle \cdot, \cdot \rangle$, is supposed to be conjugate linear in the first variable and such that the linear span of $\{\langle \xi, \eta \rangle, \xi, \eta \in X\}$ is σ -weakly dense in N . The von Neumann algebra of all N -linear continuous operators from X to X will be denoted by $\mathcal{L}_N(X)$ (or $\mathcal{L}(X)$ when $N = \mathbb{C}$). Following ([BDH], Def. 2.1), by a $M - N$ *correspondence* we mean a pair (X, π) where X is as above, and π is a unital normal faithful homomorphism from M into $\mathcal{L}_N(X)$. More briefly, such a correspondence will be denoted by X and we shall often write $x\xi$ instead of $\pi(x)\xi$.

These two notions of correspondences are related in the following way. Consider a $M - N$ correspondence X and let $\mathcal{H}(X) = X \otimes_N L_2(N)$ be the Hilbert space obtained by inducing the standard representation of N up to M via X ([Rie74], Th. 5.1). Then the left action of M and the right action

of N defined on $\mathcal{H}(X)$ by

$$x(\xi \otimes h)y = x\xi \otimes hy, \quad \text{for } \xi \in X, h \in L_2(N), x \in M, y \in N,$$

turn $\mathcal{H}(X)$ into a correspondence between these algebras.

Conversely, given a correspondence \mathcal{H} between M and N , let $X(\mathcal{H}) = \text{Hom}_{N^{\text{op}}}(L_2(N), \mathcal{H})$ be the space of continuous N^{op} -linear operators from $L_2(N)$ into \mathcal{H} . Let N acts on the right of $X(\mathcal{H})$ by composition of operators and define on $X(\mathcal{H})$ a N -valued inner product by $\langle r, s \rangle = r^*s$ for $r, s \in X(\mathcal{H})$. Then $X(\mathcal{H})$ is a self-dual Hilbert N -module ([Rie74], Th. 6.5). Moreover, M acts on the left of $X(\mathcal{H})$ by composition of operators, and we obtain in this way an $M - N$ correspondence.

The maps $X \rightarrow \mathcal{H}(X)$ and $\mathcal{H} \rightarrow X(\mathcal{H})$ are inverse to each other ([BDH], Th. 2.2 and [Rie74], Prop. 6.10), up to unitary equivalence. We shall not make any distinction between equivalent correspondences. Also, we shall often identify a correspondence \mathcal{H} and its self-dual module version $X(\mathcal{H})$. It will be useful to note that, when replacing X by $\mathcal{H} = \mathcal{H}(X) = X \otimes_N L_2(N)$, the von Neumann algebra $\mathcal{L}_N(X)$ is canonically identified with $\mathcal{L}_{N^{\text{op}}}(\mathcal{H})$, by the map which carries $x \in \mathcal{L}_N(X)$ onto the element (still denoted by x) of $\mathcal{L}_{N^{\text{op}}}(\mathcal{H})$ defined as follows:

$$x(\xi \otimes h) = (x\xi) \otimes h, \quad \text{for } \xi \in X, h \in L_2(N).$$

Let us recall from [Rie74] that two von Neumann algebras M and N are *Morita equivalent* if there exists a $M - N$ correspondence X (or equivalently a correspondence \mathcal{H} from M to N) such that M is isomorphic to $\mathcal{L}_N(X)$ (or $\mathcal{L}_{N^{\text{op}}}(\mathcal{H})$). This amounts to saying that there is a type I factor F and a projection e in $F \otimes N$ with central support 1 such that M and the reduced von Neumann algebra $e(F \otimes N)e$ are isomorphic.

Let us point out now that most of the familiar techniques used in von Neumann algebras theory and Hilbert spaces theory apply also when we work with a self-dual Hilbert N -module X . The ultrastrong topology has a useful analogue on X called the s-topology (see [BDH], §1.3). It is the topology defined on X by the family of semi-norms q_φ where φ is a normal positive form on N and

$$q_\varphi(\eta) = \varphi(\langle \eta, \eta \rangle)^{1/2}, \quad \text{for } \eta \in X.$$

Let \mathcal{H} be a correspondence from M to N , and let $\bar{\mathcal{H}}$ be the conjugate Hilbert space. If $h \in \mathcal{H}$,

we denote by \bar{h} the vector h when viewed as an element of \mathcal{H} . Then \mathcal{H} has a natural structure of correspondence from N to M by

$$y\bar{h}x = x^*\bar{h}y^*, \quad \text{for } x \in M, y \in N, h \in \mathcal{H},$$

(see [Pop], 1.3.7). It is called the adjoint or conjugate correspondence of \mathcal{H} . Notice also that with a $M - N$ correspondence $X = X(\mathcal{H})$ is associated its adjoint $\bar{X} = X(\bar{\mathcal{H}})$. But in general, there is no explicit description of \bar{X} from X .

A *subcorrespondence* of \mathcal{H} is a Hilbert subspace \mathcal{K} of \mathcal{H} , stable by the left M -action and the right N -action. In the self-dual version, a subcorrespondence Y of X is a submodule of X closed in the s -topology and stable by the left action of M . In this case, we shall say that \mathcal{K} is contained in \mathcal{H} (or that Y is contained in X), and we shall write $\mathcal{K} \leq \mathcal{H}$ (or $Y \leq X$).

Consider now three von Neumann algebras M, N, P , an $M - N$ correspondence (X, π) and an $N - P$ correspondence (Y, π_1) . We denote by $X \otimes_N Y$ the self-dual completion of the algebraic tensor product $X \otimes Y$ endowed with the obvious right action of P and the P -valued inner product

$$\langle \xi \otimes \eta, \xi_1 \otimes \eta_1 \rangle_P = \langle \eta, \langle \xi, \xi_1 \rangle_N \rangle_P, \quad \text{for } \xi, \xi_1 \in X, \eta, \eta_1 \in Y.$$

Then there is a canonical homomorphism from $\mathcal{L}_N(X)$ into $\mathcal{L}_P(X \otimes_N Y)$, sending $x \in \mathcal{L}_N(X)$ to the map $\xi \otimes \eta \mapsto (x\xi) \otimes \eta$. Moreover this homomorphism is faithful when $\pi_1 : N \rightarrow \mathcal{L}_P(Y)$ is faithful ([AD90], Lemma 1.5). By composition of this homomorphism with π , we get a left action of M into $X \otimes_N Y$ which turns $X \otimes_N Y$ into an $M - P$ correspondence, called the composition correspondence of X by Y . Put $\mathcal{H} = \mathcal{H}(X)$ and $\mathcal{K} = \mathcal{H}(Y)$. When an auxiliary faithful weight ν has been chosen on N , A. Connes has shown [Con] how to define the composition $\mathcal{H} \otimes_\nu \mathcal{K}$ of the correspondences \mathcal{H} and \mathcal{K} (see also [S]). It follows from ([S, Prop. 2.6]) that, up to equivalence, the result does not depend on the choice of ν , so we shall use the notation $\mathcal{H} \otimes_N \mathcal{K}$ instead of $\mathcal{H} \otimes_\nu \mathcal{K}$. It is easily checked that $X(\mathcal{H} \otimes_N \mathcal{K}) = X \otimes_N Y$ and thus there is no ambiguity on the notion of composition of correspondences.

Let N be a von Neumann subalgebra of a von Neumann algebra M (and then we will say that $N \subset M$ is a pair of von Neumann algebras). The Hilbert space $L_2(M)$ has a natural structure of correspondence from M to N by restricting to N the right action of M . This object, which is crucial in the study of the inclusion, is called the *standard correspondence* associated with the pair $N \subset M$ and denoted by ${}_M L_2(M)_N$. In a similar way, we may consider the correspondence ${}_N L_2(M)_M$ from N

to M , and by using J_M it is easily checked that it is equivalent to the conjugate of ${}_M L_2(M)_N$. Let us remark that any correspondence ${}_M \mathcal{H}_N$ is isomorphic to the correspondence ${}_M L_2(M) \otimes_M \mathcal{H}_N$, and to ${}_M \mathcal{H} \otimes_N L_2(N)_N$ as well.

Let \mathcal{H} be a correspondence from M to M . Then ${}_N L_2(M) \otimes_M \mathcal{H} \otimes_M L_2(M)_N$ is the correspondence from N to N obtained by restricting to N the left and right actions of M on \mathcal{H} . It will be called the restriction of H from M to N .

Let \mathcal{H} be now a correspondence from N to N . Then ${}_M L_2(M) \otimes_N \mathcal{H} \otimes_N L_2(M)_M$ is a correspondence from M to M . We will say that it is the correspondence induced by \mathcal{H} from N up to M and will denote it by $\text{Ind}_N^M \mathcal{H}$.

Let us recall now that a correspondence \mathcal{H} from a von Neumann algebra M to a von Neumann algebra N is nothing else than a representation of the binormal tensor product $M \otimes_{\text{bin}} N^{\text{op}}$ (see [EL] for the definition of the norm bin). Furthermore two correspondences are isomorphic if and only if they are unitarily equivalent when considered as representations of $M \otimes_{\text{bin}} N^{\text{op}}$. Thus every notion which makes sense for representations of C^* -algebras can also be defined for correspondences. In particular the topology defined by Fell ([Fel62], Section 1) on the space of (equivalence classes of) representations of the C^* -algebra $M \otimes_{\text{bin}} N^{\text{op}}$ gives rise to the following topology on the set $\text{Corr}(M, N)$ of (equivalence classes of) correspondences from M to N (as usual this set is restricted suitably in order to avoid paradoxically huge sets). Let $\mathcal{H}_0 \in \text{Corr}(M, N)$, $\varepsilon > 0$, $E \subset M$ and $F \subset N$ two finite sets, and $S = (h_1, \dots, h_p)$ a finite subset of H_0 . We define by $U(\mathcal{H}_0; \varepsilon, E, F, S)$ the set of $\mathcal{H} \in \text{Corr}(M, N)$ such that there exist $k_1, \dots, k_p \in H$ with

$$|\langle k_i, x k_j y \rangle - \langle h_i, x h_j y \rangle| < \varepsilon, \quad \text{for all } x \in E, y \in F, i, j = 1, \dots, p.$$

Then $\text{Corr}(M, N)$ is equipped with the well defined topology having these U 's as a basis of neighbourhoods.

If we regard correspondences as self-dual Hilbert modules, the topology may be described as follows (see [AD90], §1.12). Let $X_0 = X(\mathcal{H}_0)$, Ω a σ -weak neighbourhood of 0 in N , E a finite subset of M , and $S = \xi_1, \dots, \xi_p$ a finite subset of X_0 be given. We denote by $V(X_0; \Omega, E, S)$ (or, more briefly $V(\Omega, E, S)$) the set of correspondences X such that there exist $\eta_1, \dots, \eta_p \in X$ with

$$\langle \eta_i, x \eta_j \rangle - \langle \xi_i, x \xi_j \rangle \in \Omega, \quad \text{for all } x \in E, i, j = 1, \dots, p.$$

Then such V 's constitute a basis of neighbourhoods of X_0 in $\text{Corr}(M, N)$. Moreover, if X_0 has a cyclic vector ξ_0 , it has a basis of neighbourhoods of the form $V(X_0; \Omega, E) = V(X_0; \Omega, E, \{\xi_0\})$. Note that in this case, X_0 belongs to the closure of $X \in \text{Corr}(M, N)$ if and only if there is a net (ξ_i) in X such that $\lim_i \langle \xi_i, x \xi_i \rangle = \langle \xi_0, x \xi_0 \rangle$ σ -weakly for all $x \in M$. If $\xi \in X$, we say that the normal completely positive map $x \mapsto \langle \xi, x \xi \rangle$ from M to N is a *coefficient* of X .

In particular, the identity correspondence of M belongs to the closure of a $M - M$ correspondence X if and only if there is a net (ϕ_i) of coefficients of X such that $\lim_i \phi_i(x) = x$ σ -weakly for all $x \in M$.

Let \mathcal{H}_0 and \mathcal{H}_1 be two correspondences from M to N , and denote by π_0 and π_1 the associated representations of $M \otimes_{\text{bin}} N^{\text{op}}$. We say that \mathcal{H}_0 is *weakly contained* in \mathcal{H}_1 , and we write $\mathcal{H}_0 \prec \mathcal{H}_1$, if the representation π_0 is weakly contained in π_1 , that is if $\text{Ker } \pi_0 \supset \text{Ker } \pi_1$, or equivalently, if $\|\pi_0(x)\| \leq \|\pi_1(x)\|$ for all $x \in M \otimes N^{\text{op}}$. This amounts to saying that \mathcal{H}_0 belongs to the closure of the set of finite direct sums of copies of \mathcal{H}_1 in $\text{Corr}(M, N)$ (see [Fel62], Th. 1.1). For instance, the identity correspondence of M is weakly contained in a $M - M$ correspondence X if and only if there exists a net (ϕ_i) of completely positive maps from M to M , each of which is a finite sum of coefficients of X , such that $\lim_i \phi_i(x) = x$ for all $x \in M$. Note that we may replace the net (ϕ_i) by a bounded one by a convex combination argument (see Lemma 2.2 of [ADH]).

Definition 4.1.1. *We say that a correspondence \mathcal{H} from M to N is left amenable if ${}_M L_2(M)_M \prec {}_M \mathcal{H} \otimes_N \bar{\mathcal{H}}_M$. We say that \mathcal{H} is right amenable if $\bar{\mathcal{H}}$ is left amenable, that is if $L_2(N) \prec \bar{\mathcal{H}} \otimes_M \mathcal{H}$.*

Put $N_1 = \mathcal{L}_{N^{\text{op}}}(\mathcal{H})$. Sauvageot has proved ([Sau83], Prop. 3.1) that $\mathcal{H} \otimes_N \bar{\mathcal{H}}$ is a standard form for N_1 and thus we have

$${}_M \mathcal{H} \otimes_N \bar{\mathcal{H}}_M = {}_M L_2(N_1)_M = {}_M L_2(N_1) \otimes_{N_1} L_2(N_1)_M.$$

Therefore we see that ${}_M \mathcal{H}_N$ is left amenable if and only if ${}_M L_2(N_1)_{N_1}$ is left amenable, and thus it would be enough to study the case of an inclusion $M \subset N_1$ with $\mathcal{H} = {}_M L_2(N_1)_{N_1}$.

Now without ambiguity, we assume $N \subset M$. Then the left amenability of $N - M$ correspondence ${}_N L_2(M)_M$ is equivalent to the weak containment ${}_N L_2(N)_N \prec {}_N L_2(M) \otimes_M L_2(M)_N$, which is the condition (1) in Theorem 4.0.11. In [AD95], Anantharaman-Delaroche shows the implication (1) \Rightarrow (2) \Rightarrow (3) in Theorem 4.0.11. The key ingredient is the following proposition in [AD95].

Proposition 4.1.2. *For $N \subset M$ von Neumann algebras, the following are equivalent.*

1. ${}_N L_2(N)_N \prec {}_N L_2(M)_N$.

2. There exists a net (ϕ_i) of completely positive maps from M to N , of the form $\phi_i(x) = \sum_j (W_j^i)^* x W_j^i$, where the sum is finite and $W_j^i \in \text{Hom}_{N^{\text{op}}}(L_2(N), L_2(M))$, such that $\lim_i \phi_i(x) = x$ σ -weakly for all $x \in M$.

4.1.2 Local reflexivity

The main tool for proving the missing implication (3) \Rightarrow (1) in the Theorem 4.0.11 is the following theorem of local reflexivity.

Theorem 4.1.3. *Let N be a von Neumann algebra. For arbitrary finite dimensional subspaces $E \subset N^*$ and $F \subset N$, and arbitrary $\varepsilon > 0$, there exists a map $u : E \rightarrow N_*$ such that*

1. $\|u\|_{cb} \leq 1 + \varepsilon$;
2. $\|u^{-1}\|_{cb} \leq 1 + \varepsilon$;
3. $|\langle u(e), f \rangle - \langle e, f \rangle| \leq \varepsilon \|e\| \|f\|$.

Directly applying the local reflexivity, we have the following corollary on ultraproduct of a von Neumann algebra.

Corollary 4.1.4. *There exists an ultrafilter \mathcal{U} on some index set I , and a map $u : N^* \rightarrow \Pi_{\mathcal{U}} N_*$, such that the composition of maps $N^* \xrightarrow{u} \Pi_{\mathcal{U}} N_* \xrightarrow{\text{Lim}} N^*$, where Lim is the limit over the ultrafilter \mathcal{U} , is the identity on N^* , and $u^* : (\Pi_{\mathcal{U}} N_*)^* \rightarrow N^{**}$ is complete positive.*

Proof. For each tuple (E, F, ε) as in Theorem 4.1.3, we have a map $u_{(E, F, \varepsilon)}$. Now define the map $u(x) = (u_{(E, F, \varepsilon)}(x))_{x \in E}$. Then $\text{Lim}_{(E, F, \varepsilon)} u_{(E, F, \varepsilon)}(x)(y) = (x)(y)$ for $x \in N^*$ and $y \in N$. By Theorem 4.1.3, u is completely contractive. Hence $u^* : (\Pi_{\mathcal{U}} N_*)^* \rightarrow N^{**}$ is completely contractive and $u^*(1) = 1$. Therefore u^* is completely positive. \square

Remark 4.1.5. We have the following facts from [Ray02]:

1. $(\text{Lim})^*(x) = (x)^\circ$ for $x \in N$, and $(x)^\circ$ is a representative in $\Pi_{\mathcal{U}} N$.
2. $L_p((\Pi_{\mathcal{U}} N_*)^*) = \Pi_{\mathcal{U}} L_p(N)$.

4.1.3 Basic construction and relative amenability

Consider a tracial von Neumann algebra (M, τ) equipped with a faithful normal tracial state τ . Let $A \subset M$ be a von Neumann subalgebra in M . Then, the conditional expectation E_A can be viewed

as the orthogonal projection e_A from $L_2(M)$ onto $L_2(A) \subset L_2(M)$. It satisfies $e_A x e_A = E_A(x) e_A$ for every $x \in M$. The *basic construction* $\langle M, e_A \rangle$ is the von Neumann subalgebra $\mathbb{B}(L_2(M))$ generated by M and e_A . We note that $\langle M, e_A \rangle$ coincides with the commutant of the right A -action in $\mathbb{B}(L_2(M))$. The linear span of $\{x e_A y : x, y \in M\}$ is an ultraweakly dense $*$ -subalgebra in $\langle M, e_A \rangle$ and the basic construction $\langle M, e_A \rangle$ comes together with the faithful normal semi-finite trace Tr such that $\text{Tr}(x e_A y) = \tau(xy)$. See Section 1.3 in [Pop06] for more information on the basic construction.

Recall that for von Neumann algebras $B \subset M$, a state φ on M is said to be B -central if $\varphi \circ \text{Ad}(u) = \varphi$ for any $u \in \mathcal{U}(B)$, or equivalently if $\varphi(ax) = \varphi(xa)$ for all $a \in B$ and $x \in M$. M is *amenable* if there exists an M -central state on $\mathbb{B}(L_2(M))$, whose restriction to M equals to τ . Connes's fundamental theorem in [Con76] says that a tracial von Neumann algebra M is amenable if and only if M is *hyperfinite*, i.e. M admits an increasing net of finite-dimensional von Neumann subalgebras whose union is weakly dense in M . Also, M is amenable if and only if the identity correspondence (or trivial bimodule) ${}_M L_2(M)_M$ is weakly contained in the coarse bimodule ${}_M(L_2(M) \otimes L_2(M))_M$.

In [OP10], Ozawa and Popa adapted Connes characterization of amenable von Neumann algebras to the relative situation.

Definition 4.1.6 ([OP10]). *Let (M, τ) be a tracial von Neumann algebra and let A, B be von Neumann subalgebras of M . We say that B is amenable relative to A inside M , if the von Neumann algebra $\langle M, e_A \rangle$ admits a B -central positive functionals whose restriction to M is τ .*

Similarly Popa and Vaes in [PV14] adapted the notion of left amenability in a relative sense.

Definition 4.1.7 ([PV14]). *Let (M, τ) and (A, τ) be tracial von Neumann algebras, and $B \subset M$. An $M - A$ correspondence ${}_M \mathcal{H}_A$ is left B -amenable if the $M - B$ correspondence ${}_M L_2(M)_B$ is weakly contained in the $M - B$ correspondence $\mathcal{H} \otimes_A \bar{\mathcal{H}}$.*

Note that the above definition generalizes the notion of left amenability in Definition 4.1.1. More precisely, an $M - A$ correspondence ${}_M \mathcal{H}_A$ is left M -amenable in the sense of Definition 4.1.7, if and only if ${}_M \mathcal{H}_A$ is left amenable in the sense of Definition 4.1.1.

By definition, for $A \subset M$ and $B \subset M$, we have that B is amenable relative to A inside M , if and only if $M - A$ correspondence ${}_M L_2(M)_A$ is left B -amenable.

A characterization for the notion of relative amenability is proved in [OP10] and also in [PV14].

Theorem 4.1.8. *Let $A, B \subset M$ be finite von Neumann algebras. Then the following are equivalent:*

1. B is amenable relative to A inside M , i.e. there exists a B -central state φ on $\langle M, e_A \rangle$, such that $\varphi|_M = \tau$.
2. There exists a B -central state φ on $\langle M, e_A \rangle$, such that φ is normal on M , and faithful on $\mathcal{L}(A' \cap M)$.
3. There exists a conditional expectation Φ from $\langle M, e_A \rangle$ onto B such that $\Phi|_M = \text{id}_B$.
4. There exists a net (ξ_n) in $L_2\langle M, e_A \rangle$, such that $\lim_n \langle x\xi_n, \xi_n \rangle = \tau(x)$ for every $x \in M$ and that $\lim \| [u, \xi_n] \| = 0$ for every $u \in B$.
5. ${}_M L_2(M)_A$ is left B -amenable, i.e. ${}_M L_2(M)_B \prec {}_M L_2(M) \otimes_A L_2(M)_B$.

We will extend the definition and the equivalence theorem by [OP10] to a general non-finite von Neumann algebra N .

4.2 Proof of the main theorem

The key ingredient for proving the main theorem is the following proposition.

Proposition 4.2.1. *Let M be a von Neumann algebra. Then ${}_M L_2(M^{**})_M \prec {}_M L_2(M)_M$.*

To prove the above proposition, we need to set up the following lemma.

Lemma 4.2.2. *Let M_1 be a von Neumann subalgebra of M_2 , with a normal (non-faithful) conditional expectation $E : M_2 \rightarrow M_1$. Then there exist maps $L_p(M_1) \xrightarrow{j_p} L_p(M_2) \rightarrow L_p(M_1)$, where j_p is a complete isometry and an M_1 -bimodule map.*

Proof. For conditional expectation $E : M_2 \rightarrow M_1$, let $\mathbb{E} = E^*|_{M_{1*}} : M_{1*} \rightarrow M_2^*$.

Case I: M_2 is σ -finite and E is faithful. The case is solved in [JX08], by defining

$$j_p(d^{\frac{1-\theta}{p}} a d^{\frac{\theta}{p}}) = D^{\frac{1-\theta}{p}} a D^{\frac{\theta}{p}} \quad (4.2.1)$$

for $a \in M_1$, where $d \in M_{1*}$ is a density and $D = \mathbb{E}(d)$.

Case II: M_2 is σ -finite and E is not faithful. Let e be the support of E , and $\pi(x) = exe$ for $x \in M_1$.

We claim that $[\pi(x), e] = 0$, and therefore π is a $*$ -homomorphism. Indeed, $E(\pi(x^*)e\pi(x)) = x^*E(e)x = x^*x = E(\pi(x)^*\pi(x))$. Hence $e\pi(x)^*e\pi(x)e = e\pi(x)^*\pi(x)e$, and therefore $e\pi(x)e = \pi(x)e$.

Let $D = \mathbb{E}(d)$, the $D \in L_p(eM_2e) = eL_p(M_2)e \subseteq L_p(M_2)$. Then j_p defined in (4.2.1) is a M_1 -bimodule map. Indeed, since π is a $*$ -homomorphism, $j_p(ad^{\frac{1}{p}}) = aD^{\frac{1}{p}} = aeD^{\frac{1}{p}} = eaeD^{\frac{1}{p}} \in eL_p(M_2)e$.

Case III: M_2 is not σ -finite. Recall that for a weight ϕ on M , there exists a net of normal strict semifinite faithful weights (ϕ_i) , such that $\phi = \sum_{i \in I} e_i \phi_i e_i$, where e_i 's are mutually orthogonal projections, and $\text{supp}(\phi_i) = e_i$. For any finite subset $F \subset I$, let $e_F = \sum_{i \in F} e_i$. Case II implies that $e_F M_1 e_F \subseteq e_F M_2 e_F$, and j_p maps $L_p(e_F M_1 e_F)$ to $L_p(e_F M_2 e_F)$, consistently for the order of subsets. Note that $e_F \rightarrow 1$ in the strong operator topology. Therefore $\cup_F e_F L_p(M_2) e_F^{-\|\cdot\|} = L_p(M_2)$. \square

Proof of Proposition 4.2.1. Use Lemma 4.2.2 for $M_1 = M^{**}$ and $M_2 = (\Pi_{\mathcal{U}} M_*)^*$. By Theorem 4.1.3 and its corollary, there exists $u^* : (\Pi_{\mathcal{U}} M_*)^* \rightarrow M^{**}$ which is a $*$ -homomorphism, and hence a conditional expectation. By Lemma 4.2.2, there exists $j_2 : L_2(M^{**}) \rightarrow L_2(\Pi_{\mathcal{U}} M_*)^* = \Pi_{\mathcal{U}} L_2(M)$. By the definition of weak containment, we have naturally that ${}_M \Pi_{\mathcal{U}} L_2(M)_M \prec {}_M L_2(M)_M$. Therefore ${}_M L_2(M^{**})_M \prec {}_M L_2(M)_M$. \square

Applying Proposition 4.2.1 in particular for von Neumann algebras $N \subset M$, we have that ${}_N L_2(M^{**})_N \prec {}_N L_2(M)_N$. Now we are ready to prove the missing implication in the main theorem.

Proof of Theorem 4.0.11. (3) \Rightarrow (1) in Theorem 4.0.11: Let $E : M \rightarrow N$ be the conditional expectation. Let $E_* = E^*|_{N_*}$ and $\mathcal{E} = (E_*)^* : M^{**} \rightarrow N^{**}$ be the normal conditional expectation. Then by Lemma 4.2.2 and Proposition 4.2.1, we have that ${}_N L_2(N)_N \subset {}_N L_2(N^{**})_N \rightarrow {}_N L_2(M^{**})_N \prec {}_N L_2(M)_N$. \square

4.3 Relative Amenability

In this section, we will generalize Theorem 4.1.8 to a general setting.

Theorem 4.3.1. *Let N be a von Neumann algebra (might not be finite) and A, B be von Neumann subalgebras of N such that there exists a normal conditional expectation $E_B : N \rightarrow B$. Let $\mathcal{N} = \mathbb{B}(L_2(N)) \cap (A^{\text{op}})'$. Then the following statements are equivalent:*

1. *there exists a conditional expectation $E : \mathcal{N} \rightarrow B$ (non-normal), such that $E|_N = E_B$.*
2. *B is amenable inside N over A , i.e. ${}_N L_2(N)_B \prec {}_N L_2(N) \otimes_A L_2(N)_B$;*

Remark 4.3.2. Recall that in [Sau83], suppose $M \subseteq \mathbb{B}(\mathcal{H})$ is a von Neumann algebra and $A = (M')^{\text{op}}$. Then we have ${}_M \mathcal{H} \otimes_A \bar{\mathcal{H}}_M = L_2(M)$. Now let $\mathcal{H} = L_2(N)$, $A \subseteq N$, and $\mathcal{N} = (A^{\text{op}})' \subseteq \mathbb{B}(L_2(N))$.

Therefore we have ${}_N L_2(N) \otimes_A L_2(N)_N = L_2(\mathcal{N})$. In particular, if A is conditioned in N , such that there exists a projection $e_A : L_2(N) \rightarrow L_2(A)$, then $L_2(N) \otimes_A L_2(N) = L_2(\langle N, e_A \rangle)$. Hence our result generalized Theorem 4.1.8 in a non- II_1 setting.

Proof of Theorem 4.3.1. (1) \Rightarrow (2): Suppose we have a conditional expectation $E : \mathcal{N} \rightarrow B$, and let $E_* = E^*|_{B_*} : B_* \rightarrow \mathcal{N}^*$. Then $\mathcal{E} = (E_*)^* : \mathcal{N}^{**} \rightarrow B^{**}$ is a normal conditional expectation. By local reflexivity and Lemma 4.2.2, we have

$$L_2(B) \subset L_2(B^{**}) \xrightarrow{j_2^\mathcal{E}} L_2(\mathcal{N}^{**}) \xrightarrow{j_2^{LR}} \Pi_{\mathcal{U}} L_2(\mathcal{N}) = \Pi_{\mathcal{U}}(L_2(N) \otimes_A L_2(N)).$$

We claim that $N j_2^\mathcal{E}(L_2(B)) = L_2(N)$, and therefore ${}_N L_2(N)_B$ is contained in ${}_N L_2(\mathcal{N}^{**})_B$.

Indeed, for the simplicity of writing, let Tr denote the formal Haagerup trace induced from the inner product. For b_1, b_2 in B , and n_1, n_2 in N , we have

$$\begin{aligned} \langle n_1 j_2^\mathcal{E}(b_1), n_2 j_2^\mathcal{E}(b_2) \rangle &= \text{Tr}(j_2^\mathcal{E}(b_1)^* n_1^* n_2 j_2^\mathcal{E}(b_2)) = \text{Tr}(n_1^* n_2 j_2^\mathcal{E}(b_2) j_2^\mathcal{E}(b_1)^*) \\ &= \text{Tr}(n_1^* n_2 E_*(b_2 b_1^*)) = \text{Tr}(E(n_1^* n_2) b_2 b_1^*) = \text{Tr}(n_1^* n_2 b_2 b_1^*) = \langle n_1 b_1, n_2 b_2 \rangle. \end{aligned}$$

Hence by Lemma 4.2.1, we have ${}_N L_2(N)_N \subseteq {}_N L_2(\mathcal{N}^{**})_B \prec {}_N L_2(\mathcal{N})_B$.

(2) \Rightarrow (1): Let d be a normal faithful state on B , i.e. a density in $L_1(B)$. Then there exists a net $\xi_n \in L_2(N) \otimes_A L_2(N)$, such that for b_1, b_2 in B , and n_1, n_2 in N ,

$$\langle n_1 \xi_n b_1, n_2 \xi_n b_2 \rangle \rightarrow \langle n_1 d^{\frac{1}{2}} b_1, n_2 d^{\frac{1}{2}} b_2 \rangle = \text{Tr}(b_1^* d^{\frac{1}{2}} n_1^* n_2 d^{\frac{1}{2}} b_2) = \text{Tr}(d^{\frac{1}{2}} b_2 b_1^* d^{\frac{1}{2}} n_1^* n_2).$$

Define $\Phi : \mathcal{N} \rightarrow B^{\text{op}*}$ by

$$\Phi(x)(b) = \lim_n \langle \xi_n, b \xi_n x \rangle = \lim_n \text{Tr}(\xi_n^* b \xi_n x).$$

Note that $\Phi(1)(b) = \text{Tr}(\xi_n \xi_n^* b) \rightarrow \text{Tr}(b)$, we have $\Phi(1) \in B_*$. Therefore $\Phi : \mathcal{N} \rightarrow B_*^{\text{op}}$. By a standard construction, there exists $E : \mathcal{N} \rightarrow B$, such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\Phi} & B_*^{\text{op}} \\ & \searrow E & \uparrow d^{\frac{1}{2}} \cdot d^{\frac{1}{2}} \\ & & B \end{array}$$

It is easy to check from the construction that $E|_N = E_B$. □

Remark 4.3.3. From the proof of (2) \Rightarrow (1), the statements are equivalent to

(2') There exists a net $\xi_n \in L_2(N) \otimes_A L_2(N)$ with norm 1, such that

$$\langle \xi_n, b\xi_n x \rangle \rightarrow \langle d^{\frac{1}{2}}, bd^{\frac{1}{2}}x \rangle.$$

Actually they are all equivalent to the following conditions similar to Theorem 4.1.8.

(2'') For density $d \in B_*$, and $x \in \mathcal{N}$ and $b \in B$, there exists ξ_n , such that the following hold:

- (a) $\langle \xi_n, \xi_n x \rangle \rightarrow \text{Tr}(dx)$;
- (b) $\|\xi_n b - \sigma_t^d(b)\xi_n\|_2 \rightarrow 0$, where σ_t^d denote the one parameter modular automorphism group associate with normal faithful state d .

Apparently we have (2'') \Rightarrow (2') \Rightarrow (1). For (1) \Rightarrow (2''), (a) follows from the proof of the theorem. For (b), since $L_2(B) \rightarrow \Pi_{\mathcal{U}}(L_2(N) \otimes_A L_2(N))$, we have $\|bd^{\frac{1}{2}} - \sigma_t^d(b)d^{\frac{1}{2}}\|_2 = 0$. Hence $\|\xi_n b - \sigma_t^d(b)\xi_n\|_2 \rightarrow 0$.

4.4 Relation with relatively weak injectivity

In this section, we examine the relation between relatively weak injectivity and the weak containment of correspondences. Recall the notion of relatively weak injectivity from Kirchberg [Ki93]. For C^* -subalgebra $A \subseteq B$, we say that A is *relatively weakly injective* in B , if there exists a u.c.p. map $\varphi : B \rightarrow A^{**}$, such that $\varphi|_A = \text{id}_A$.

Let C^* -algebra A be a subalgebra of a von Neumann algebra M , and let $N = A'' \cap M \subseteq M$. Consider the following stronger statement than relative weak injectivity:

- (*) If A is relatively weakly injective in M , then there exists a u.c.p. map $\Phi : M \rightarrow N$ such that $\Phi|_N = \text{id}_N$.

In the first glance, the statement seems to be natural. Since $A'' = N$, there exists a central projection z in A^{**} , such that $N = zA^{**}$. Hence by the relative weak injectivity of $A \subseteq M$, we have a composition $\Phi : M \rightarrow A^{**} \rightarrow N$ of the weak expectation and the projection. It is obvious that $\Phi|_A = \text{id}_A$. However it is not clear that such map restricted on N is the identity. In fact, the above statement (*) does not hold necessarily, which means there does exist an *r.w.i.* inclusion $A \subset M$, such

that we cannot have such Φ . A trivial example is when $M = \mathbb{B}(\mathcal{H}_u)$ is the universal representation of A , then $N \simeq A^{**}$. The existence of Φ implies injectivity of A^{**} , which is equivalent to nuclearity of A . Hence we would have the WEP implies nuclearity.

In this section, we will construct a more explicit counter example to see some interesting implications. First, we establish the following Swap Lemma.

Lemma 4.4.1. *[Swap Lemma] Assume (*) is true. Let π_0 and π_1 be disjoint representations of A in M_0 and M_1 respectively, and π_0 is faithful. If $\pi_0(A) \subseteq M_0$ is relatively weakly injective, then there exists a conditional expectation $\psi : M_1 \rightarrow \pi_1(A)''$.*

Proof. Let $N_0 = \pi_0(A)'' \cap M_0$ and $N_1 = \pi_1(A)'' \cap M_1$. Let $M = M_0 \oplus M_1$ and π be the representation of A on M induced from π_0 and π_1 , and $N = \pi(A)'' \cap M$.

First, we claim that $\pi(A)$ is relatively weakly injective in M . Indeed, suppose P_0 is the projection from M to M_0 , and φ_0 is the weak expectation from M_0 to $\pi_0(A)^{**}$. Since π_0 is faithful, so is π , we have that

$$\pi(A) \xrightarrow{\pi_0 \circ \pi^{-1}} \pi_0(A) \xrightarrow{\pi^{-1} \circ \pi_0} M \xrightarrow{P_0} M_0 \xrightarrow{\varphi_0} \pi_0(A)^{**} \xrightarrow{(\pi \circ \pi_0^{-1})^{**}} \pi(A)^{**},$$

which is the identity on $\pi(A)$.

By (*), there exists a u.c.p. $\Phi : M \rightarrow N$, such that $\Phi|_N = \text{id}_N$.

Now let $\pi^{\text{nor}} : A^{**} \rightarrow N$, $\pi_0^{\text{nor}} : A^{**} \rightarrow N_0$ and $\pi_1^{\text{nor}} : A^{**} \rightarrow N_1$, $z_0 = \text{supp}(\pi_0^{\text{nor}})$ and $z_1 = \text{supp}(\pi_1^{\text{nor}})$. Then by disjointness of π_0 and π_1 , we have that $z_0 \perp z_1$ and $\pi^{\text{nor}} = \pi_0^{\text{nor}} \oplus \pi_1^{\text{nor}}$.

Denote $z_0^N = \pi^{\text{nor}}(z_0)$ and $z_1^N = \pi^{\text{nor}}(z_1)$. Then we have $z_0^N N = N_0$ and $z_1^N N = N_1$. We define a u.c.p. map ψ on M_1 by $\Phi(x) = \Phi(0, x)z_1^N$. It is easy to show that this is a conditional expectation from M_1 to N_1 . \square

We will construct a counter example to show (*) is not true. Let \mathbb{F}_{n+1} be the free group of $n+1$ generators g_1, \dots, g_{n+1} , and \mathbb{F}_n be the free subgroup of \mathbb{F}_{n+1} generated by g_1, \dots, g_n . We define a state φ on $C_\lambda^* \mathbb{F}_{n+1}$ by:

$$\varphi(\lambda(g)) = \begin{cases} 1 & \text{if } g \in g_{n+1}^{\mathbb{Z}}; \\ 0 & \text{else.} \end{cases} \quad (4.4.1)$$

This is a well-defined state from the composition of the conditional expectation and the trivial state $\varphi : C_\lambda^* \mathbb{F}_{n+1} \xrightarrow{E} C_\lambda^* \mathbb{Z} \rightarrow \mathbb{C}$. We claim that φ is singular with respect to the trace on $L\mathbb{F}_{n+1}$.

Lemma 4.4.2. *Let G_0 be a subgroup of G_1 , and φ_0 be a state on $C_\lambda^*G_0$, which admits a singular extension to LG_0 . Then φ_0 admits a singular extension on LG_1 .*

Proof. Let $\hat{\varphi}$ be the singular extension of φ_0 on LG_0 . Then there exists a net of increasing projections q_λ converging to 1 in strong operator topology, such that $\hat{\varphi}|_{C_\lambda^*G_0} = \lim((1 - q_\lambda)x(1 - q_\lambda))$.

Now we define a state ψ on LG_1 by $\psi(x) = \lim_{\lambda \rightarrow \infty} \hat{\varphi}((1 - q_\lambda)E(x)(1 - q_\lambda))$, where $E : LG_1 \rightarrow LG_0$ is the conditional expectation. We claim that this ψ is singular with respect to the trace on LG_1 .

Indeed, let $\psi = \psi_n + \psi_s$ be the singular decomposition of ψ . Then by the increasing property of the projections, we have

- (a) $\psi(x) = \psi((1 - q_\lambda)x(1 - q_\lambda))$ for arbitrary λ ;
- (b) $\psi_s(x) = \psi_s((1 - q_\lambda)x(1 - q_\lambda))$ for arbitrary λ ;
- (c) $\psi_n((1 - q_\lambda)x(1 - q_\lambda)) \rightarrow 0$.

Hence we have $\psi(x) = \psi_s(x)$, i.e. ψ is a singular on LG_1 . \square

Corollary 4.4.3. *The state φ defined in (4.4.1) is singular on $L\mathbb{F}_{n+1}$ with respect to the trace.*

Proof. Let $H \simeq \mathbb{Z} \simeq \mathbb{T}$ be the subgroup of \mathbb{F}_{n+1} generated by g_{n+1} , and $\varphi_n = \frac{n}{2}1_{[-1/n, 1/n]}$ be the state on $C_\lambda^*\mathbb{T}$, and $\hat{\varphi}$ be the limit of φ_n in weak*-topology. Then $\hat{\varphi}|_{C_\lambda^*\mathbb{T}} = \delta_0 \neq 0$ is singular on $L\mathbb{T}$. By the proof of Lemma 4.4.2, we can construct a singular state ψ on $L\mathbb{F}_{n+1}$ by $\psi(x) = \lim_{\lambda \rightarrow \infty} \hat{\varphi}((1 - q_\lambda)E(x)(1 - q_\lambda))$.

Now let $x = \lambda(g)$ for $g \in \mathbb{F}_{n+1}$. If $g \notin H$, from the construction of ψ , we have that $\psi(x) = 0$. If $g \in H$, then $\psi(x) = 1$. Therefore we have that $\psi|_{C_\lambda^*\mathbb{F}_{n+1}} = \varphi$, and hence φ is singular with respect to the trace on $L\mathbb{F}_{n+1}$. \square

Now we apply the Swap Lemma to disproof (*).

Proposition 4.4.4. *There exists an r.w.i. inclusion $A \subset M$ with $N = A'' \cap M$, such that there is no u.c.p. map $\Phi : M \rightarrow N$ with $\Phi|_N = \text{id}_N$.*

Proof. Suppose we have such map Φ . Then we have the Swap Lemma.

Let $\pi_\lambda : C_\lambda^*\mathbb{F}_{n+1} \rightarrow L\mathbb{F}_{n+1}$ be the faithful representation. It is well known that $C_\lambda^*\mathbb{F}_{n+1} \hookrightarrow L\mathbb{F}_{n+1}$ is r.w.i.. Let φ be the state on $C_\lambda^*\mathbb{F}_{n+1}$ defined in (4.4.1), and $\pi_\varphi : C_\lambda^*\mathbb{F}_{n+1} \rightarrow \mathbb{B}(\mathcal{H}_\varphi)$ be the GNS construction from φ . By Corollary 4.4.3, we have that π_λ and π_φ are disjoint representations. By the Swap lemma, there exists a conditional expectation $\psi : \mathbb{B}(\mathcal{H}_\varphi) \rightarrow \pi_\varphi(C_\lambda^*\mathbb{F}_{n+1})''$.

Now let $F^+ = \{g_{i_1}^{k_1} \cdots g_{i_l}^{k_l} \mid i_l \neq n+1\}$ be the subset of words in \mathbb{F}_{n+1} , which do not end with g_{n+1} . We claim that $\mathcal{H}_\varphi = \ell_2(F^+)$. Indeed, suppose $w_1 = vv'$ and $w_2 = vv''$ in F^+ . If $w_1 \neq w_2$, then we have either $v' \neq 0$ or $v'' \neq 0$. Without loss of generality, let $v' \neq 0$, then we have $w_2^{-1}w_1 = v''^{-1}v'$, and $w_1^{-1}w_2 = v'v''$ and hence $\varphi(w_2^{-1}w_1) \neq 0$, $\varphi(w_1^{-1}w_2) \neq 0$. On the other hand, for arbitrary $w \in \mathbb{F}_{n+1}$, by the construction of ψ from the conditional expectation, we have that $\varphi(\lambda(wg_{n+1}^k)) = \varphi(\lambda(w))$. Therefore $\mathcal{H}_\varphi = \ell_2(F^+)$.

Now let $F^+ = \mathbb{F}_n \times R$ be the set decomposition, where R is the right coset of \mathbb{F}_n , and $r_1 \neq r_2$ if and only if $r_1 r_2^{-1} \notin \mathbb{F}_n$. Since $\mathcal{H}_\varphi = \ell_2(F^+) = \oplus_{r \in R} \mathcal{H}_r$, we have $\mathbb{B}(\mathcal{H}_\varphi) = \mathbb{B}(\ell_2(F^+)) = \mathbb{B}(\ell_2(\mathbb{F}_n \times R)) \simeq \mathbb{B}(\ell_2(\mathbb{F}_n)) \bar{\otimes} \mathbb{B}(\ell_2(R))$, and π_φ embeds each $x \in C_\lambda^* \mathbb{F}_n$ on the diagonal in $\mathbb{B}(\ell_2(\mathbb{F}_n)) \bar{\otimes} \ell_\infty(R)$. Since R is countable, there exists a normal conditional expectation $E = \text{id} \otimes P_{\mathbb{D}} : \mathbb{B}(\ell_2(\mathbb{F}_n)) \bar{\otimes} \mathbb{B}(\ell_2(R)) \rightarrow \mathbb{B}(\ell_2(\mathbb{F}_n)) \bar{\otimes} \ell_\infty(R)$, where \mathbb{D} is the diagonal in R and $P_{\mathbb{D}}$ is the projection on the diagonal. Let $M_\varphi = \pi_\varphi(C_\lambda^* \mathbb{F}_{n+1})''$ and $M = \pi_\varphi(C_\lambda^* \mathbb{F}_n)''$. We claim that $E|_{M_\varphi}$ maps M_φ to M .

Indeed, let $g \in F^+ = \mathbb{F}_n \times R$. If $g \in \mathbb{F}_n \mathbb{D}$, where \mathbb{D} is the diagonal operator, then it is easy to check that $E(\lambda(g)) = \lambda(g)$. If $g \notin \mathbb{F}_n \mathbb{D}$, let $g_1, g_2 \in \mathbb{F}_n$, and $r_1, r_2 \in R$. Consider the matrix unit $\langle (e_{(g_2, r_2)}, E(\lambda(g))e_{(g_1, r_1)}) \rangle = \delta_{r_1, r_2} \langle e_{(g_2, r_1)}, e_{(g g_1^{-1}, r_1)} \rangle$. This is not zero if and only if $g = g_2 g_1^{-1} \in \mathbb{F}_n$. Hence we have the image of M_φ is in M .

Let P_{H_0} be the coordinate projection on $\mathbb{B}(\ell_2(\mathbb{F}_n)) \bar{\otimes} \ell_\infty(R)$. Then we have $\Psi = P_{H_0} E \psi P_{H_0} : \mathbb{B}(\mathcal{H}_\varphi) \rightarrow L\mathbb{F}_n$.

Now we have that $C_\lambda^* \mathbb{F}_n \xrightarrow{\pi_\varphi} \mathbb{B}(\mathcal{H}_\varphi) \xrightarrow{\Psi} L\mathbb{F}_n$. Notice that from [HP], we have the map $V : R_n \cap C_n \rightarrow C_\lambda^* \mathbb{F}_n$ and the map $W : L\mathbb{F}_n \rightarrow R_n \cap C_n$, where R_n is the n -dimensional row space, and C_n is the n -dimensional column space, with $\|V\|_{\text{cb}} \leq 2$ and $\|W\|_{\text{cb}} \leq 1$. Therefore we have that $R_n \cap C_n$ factors through $\mathbb{B}(\mathcal{H}_\varphi)$ with the factorization norm $\gamma_\infty(R_n \cap C_n) \leq 2$. When n is large enough, this contradicts the fact that $\gamma_\infty(R_n \cap C_n) \geq \sqrt{n}/2$ by [HP]. \square

Remark 4.4.5. Suppose $A \subset M$ is *r.w.i.*, and $N = A'' \cap M$. Consider the diagram

$$A \subset N \subset M \rightarrow A^{**} \rightarrow N$$

Let Φ be the composition map from M to N , then we have $\Phi|_A = \text{id}_A$ by *r.w.i.* property. But $\Phi|_N \neq \text{id}_N$ in full generality, i.e. there exists $x \in N$, such that $\Phi(x) \neq x$. Notice that there exist a net Φ_λ converging to Φ in point-weak* topology, and a net $a_s \subset A$ converging to x weakly. Hence we have $\Phi(x) = \lim_\lambda \Phi(x) = \lim_\lambda \lim_s \Phi_\lambda(a_s)$. On the other hand, $x = \lim_s \lim_\lambda \Phi_\lambda(a_s)$. Therefore our

result shows that the limits are not interchangeable.

Remark 4.4.6. Let $A = C_\lambda^* \mathbb{F}_{n+1}$, $M = L\mathbb{F}_{n+1} \oplus \mathbb{B}(\mathcal{H}_\varphi)$, and $N = A'' \cap M$. Let $u = (1, -1) \in M$. Then $u^2 = 1$, and hence it is a unitary, and $u \in A' \cap M$. Then the failure of the existence of Φ which is identity on N , implies that there is no such Φ , that $\Phi|_A = \text{id}_A$ and $\Phi(u) = u$. This leads to the following corollary.

Corollary 4.4.7. *There exists an r.w.i inclusion $A \subset M$ with $N = A'' \cap M$, and a unitary $u \in A' \cap N$, such that for C^* -algebra $C^*\langle A, u \rangle$ generated by A and u , there is no such map $\Phi : M \rightarrow N$ with $\Phi|_{C^*\langle A, u \rangle} = \text{id}_{C^*\langle A, u \rangle}$. In particular, for r.w.i. inclusion $A \subset M$, we have*

$$A \subset C^*\langle A, u \rangle \subset M$$

such that $C^*\langle A, u \rangle$ is not r.w.i. in M .

Remark 4.4.8. Assume the setting $A \subset M$ with $N = A'' \cap M$. Consider the following statements

- (a) There exists a u.c.p. map $\Phi : M \rightarrow N$, such that $\Phi_A = \text{id}_A$.
- (b) There exists a norm one projection $\Phi : M \rightarrow N$.

The above example shows that the statement (a) does not imply the statement (b). Let us consider another related statement.

- (c) There exist a net of normal u.c.p. maps $\Phi_t : M \rightarrow N$, such that $\Phi_t(x)$ converges to x σ -weakly for $x \in A$.

We can also show that (c) does not imply (b) either, by our example $A = C_\lambda^* \mathbb{F}_{n+1}$ and $M = L\mathbb{F}_{n+1} \oplus \mathbb{B}(\mathcal{H}_\varphi)$.

Let T_t be the Poisson semigroup on $L\mathbb{F}_{n+1}$, defined as

$$T_t : \lambda_g \mapsto e^{-t|g|} \lambda_g,$$

where $|\cdot|$ is the word length with respect to the generating set. By Haagerup's inequality [Haar79], T_t is a completely positive map $L\mathbb{F}_{n+1} \rightarrow C_\lambda^* \mathbb{F}_{n+1}$. Then we have that

$$M = L\mathbb{F}_{n+1} \oplus \mathbb{B}(\mathcal{H}_\varphi) \xrightarrow{T_t \oplus 0} C_\lambda^* \mathbb{F}_{n+1} \xrightarrow{\lambda \oplus \pi_\varphi} N.$$

Let $\Phi_t = (\lambda \oplus \pi_\varphi) \circ (T_t \oplus 0)$. Clearly it is the identity on A . However we have already shown that (b) fails in this setting $A \subset M$.

Now let us relate above result in a setting of correspondence and weak containment.

Corollary 4.4.9. *Suppose A is relatively weakly injective in M , then we have ${}_A L_2(A^{**})_A \prec {}_A L_2(M)_A$. However, if in addition let $N = A'' \cap M$, then there exists a pair of inclusions $A \subset N \subset M$, such that ${}_N L_2(A^{**})_N \not\prec {}_N L_2(M)_N$.*

Proof. By relatively weak injectivity of $A \subset M$, there exists a normal conditional expectation $M^{**} \rightarrow A^{**}$. Hence the first assertion follows from Proposition 4.2.1 that ${}_A L_2(A^{**})_A \prec {}_A L_2(M^{**})_A \prec {}_A L_2(M)_A$. For the second assertion, by Proposition 4.4.4 and Theorem 4.0.11, there exists $A \subset N \subset M$ such that ${}_N L_2(A^{**})_N \not\prec {}_N L_2(M^{**})_N$ \square

In a more general setting with $A \subset N \subset M$, where A is a C^* -algebra and $N = A'' \cap M$, consider the correspondence ${}_A \mathcal{H}_N$ as a Hilbert space with a representation $\pi : A \otimes_{\text{nor}} N \rightarrow \mathbb{B}(\mathcal{H})$. One has a similar topology on $\text{Corr}(A, N)$ described in [ADH]. Then we can establish a similar proposition for $A - N$ correspondence, as in Theorem 4.0.11.

Proposition 4.4.10. *Let A be a C^* -subalgebra of a von Neumann algebra M , and $N = A'' \cap M$. Consider the following statements:*

1. ${}_A L_2(N)_N \prec {}_A L_2(M)_N$.
2. *There exist a net of normal u.c.p. maps $\phi_i : M \rightarrow N$, such that $\phi_i(x)$ converges to x σ -weakly for $x \in A$.*
3. *There exists a u.c.p. map $\Phi : M \rightarrow N$, such that $\Phi|_A = \text{id}_A$.*

Then we have $(1) \Rightarrow (2) \Rightarrow (3)$.

Proof. $(1) \Rightarrow (2)$: As Proposition 4.1.2 is the key ingredient for showing $(1) \Rightarrow (2)$ in Theorem 4.0.11, following the same procedure of the proof in [AD95], we can show that ${}_A L_2(N)_N \prec {}_A L_2(M)_N$ if and only if there exists a net of maps (ϕ_i) with the form in Proposition 4.1.2, such that (ϕ_i) converges to the identity on A point σ -weakly.

$(2) \Rightarrow (3)$ follows from the standard accumulation argument of the net (ϕ_i) . \square

Remark 4.4.11. In general we do not have $(3) \Rightarrow (1)$ or $(3) \Rightarrow (2)$. As above the two different copies of N will lead to a counter example. Indeed, if we assume (3) , then we have a composition

$$N_* \xrightarrow{\Phi^*} M^* \rightarrow N_*.$$

Since A is weakly dense in N , there exists a $d \in L_2(N)$ such that Ad is dense in N_* , and

$$\langle \Phi^*(ad), b \rangle = \langle ad, \Phi(b) \rangle = \text{tr}(bad),$$

for a, b in A , and hence the above composition map is identity on Ad . Therefore we have a conditional expectation $(\Phi^*)^* : M^{**} \rightarrow N$. Then we have the composition

$$N \hookrightarrow M^{**} \xrightarrow{(\Phi^*)^*} N,$$

which is in general not the identity on N .

Assume we have a lifting $\pi_1 : N \rightarrow M^{**}$ of N induced from $(\Phi^*)^*$. Then we will have that

$${}_N L_2(N)_N \prec_{\pi_1(N)} L_2(M^{**})_{\pi_1(N)}. \quad (4.4.2)$$

Since the map Φ preserves the identity on A , with the help of local reflexivity, we have in particular

$${}_A L_2(N)_A \prec {}_A L_2(M^{**})_A \prec {}_A L_2(M)_A.$$

But in general we cannot fix the position of N in M^{**} , and hence we cannot replace $\pi_1(N)$ by N in (4.4.2).

For a specific counter example for $A - N$ correspondence, let us come back to the setting $A = C_\lambda^* \mathbb{F}_{n+1}$ and $M = L\mathbb{F}_{n+1} \oplus \mathbb{B}(\mathcal{H}_\varphi)$, and let $N = L\mathbb{F}_{n+1} \oplus M_\varphi$, with M_φ in the proof of Proposition 4.4.4. Then we have that

$$L_2(M) = {}_A L_2(\mathbb{F}_{n+1})_{L\mathbb{F}_{n+1}} \oplus {}_A L_2(\mathbb{B}(\mathcal{H}_\varphi))_{M_\varphi},$$

$$L_2(N) = {}_A L_2(\mathbb{F}_{n+1})_{L\mathbb{F}_{n+1}} \oplus {}_A L_2(M_\varphi)_{M_\varphi}.$$

Applying the projection on the second component, then (1) implies that

$${}_A L_2(M_\varphi)_{M_\varphi} \prec {}_A L_2(\mathbb{B}(\mathcal{H}_\varphi))_{M_\varphi},$$

which implies that M_φ is injective. This is the conclusion of the Swap Lemma, and hence leads to a contradiction as in the proof of Proposition 4.4.4.

A similar proof will work for (3) $\not\Rightarrow$ (2), by replacing $L_2(M)_{\pi_1(N)}$ by $\ell_2(L_2(M)_{\pi_i(N)})$, with π_i as π_1 constructed above.

By observing the issue of different positions, we have the following proposition.

Proposition 4.4.12. *Let A be a C^* -subalgebra of a von Neumann algebra M , with $N = A'' \cap M$. Then the following statements are equivalent.*

1. *There exists a u.c.p. $\Phi : M \rightarrow N$, such that $\Phi|_A = \text{id}_A$.*
2. *There exists a lifting $\pi_1 : N \rightarrow M^{**}$, such that ${}_A L_2(N)_N \prec {}_A L_2(M^{**})_{\pi_1(N)}$.*

Proof. The implication (1) \Rightarrow (2) follows from (4.4.2) in above remark.

Suppose we have (2). By Proposition 4.4.10, there exists a map $\psi : M^{**} \rightarrow \pi_1(N)$. Composing with π_1^{-1} , we have a map

$$\Psi : M^{**} \xrightarrow{\psi} \pi_1(N) \xrightarrow{\pi_1^{-1}} N.$$

It is easy to see that $\Psi|_A = \text{id}_A$. Now let $\Phi = \Psi|_M$, and we have the desired result. \square

Remark 4.4.13. If in addition we have an automorphism θ on M^{**} , such that $\theta(\pi_1(N)) = N$, and $\theta|_A = \text{id}_A$, then we have the implication (3) \Rightarrow (1) in Proposition 4.4.10.

Now let us summarize all the implications above.

Theorem 4.4.14. *Let $A \subset N = A'' \subset M$. Consider the following statements:*

1. *A is relatively weakly injective in M .*
2. *There exists a u.c.p. $\Phi : M \rightarrow N$, such that $\Phi|_A = \text{id}_A$;*
3. *There exists a net of normal u.c.p. $\varphi_i : M \rightarrow N$, such that $\varphi_i(x) \rightarrow x$ σ -weakly for $x \in A$;*
4. *${}_A L_2(N)_N \prec {}_A L_2(M)_N$;*
5. *there exists a norm one projection $\Phi : M \rightarrow N$.*

Then we have $(5) \Rightarrow (2)$ and (3) ; $(1) \Rightarrow (2)$; and $(4) \Rightarrow (3) \Rightarrow (2)$. Also $(2) \not\Rightarrow (4)$; $(1) \not\Rightarrow (5)$; and $(3) \not\Rightarrow (5)$.

Chapter 5

Construction of Many Tensor Norms

Tensor norms on C^* -algebras have been intensively studied by Effros, Lance, Connes, and Kirchberg, as a part of noncommutative analogue of Grothendieck's program. It turns out to be particularly important for the investigation of different C^* -norms on tensor products of C^* -algebras and the analysis of approximation properties such as nuclearity and injectivity.

Let A and B be C^* -algebras. It is always possible to put a C^* -norm on the algebraic tensor product $A \otimes B$. For example, the spatial (or minimal) tensor product norm $\|\cdot\|_{\min}$ and the maximal tensor product $\|\cdot\|_{\max}$ are always C^* -norms on $A \otimes B$. As the names suggest, the spatial tensor norm is the smallest C^* -norm one can place on $A \otimes B$, and the maximal is the largest. In general these norms do not agree. The C^* -algebra A is said to be nuclear if $A \otimes B$ admits a unique C^* -norm for all choices of C^* -algebras B , or equivalently, if $\|\cdot\|_{\min} = \|\cdot\|_{\max}$ on $A \otimes B$ for every C^* -algebra B . In 1995, Junge and Pisier [JP95] discovered that the min and max norm do not coincide on $\mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\mathcal{H})$. However, we have very limited ways of constructing norms on tensor products. Recently in [OP14], Ozawa and Pisier have demonstrated that $\mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\mathcal{H})$ admits 2^{\aleph_0} number of norms.

In this chapter, we find a simple mechanism to construct many norms on some tensor products as follows. The idea is the construction of induced norm from the subalgebra structure inspired by the work in Chapter 3. The main theorem is the following:

Theorem 5.0.15. *For a unital separable C^* -algebra A which does not have the weak expectation property, there are 2^{\aleph_0} number of distinct norms on the following tensor product:*

1. $A^{\otimes \mathbb{N}} \otimes A^{\text{op}}$, where $A^{\otimes \mathbb{N}}$ is the countably infinite tensor product of A completed in min-tensor norm;
2. $(*_\mathbb{N}(A, \varphi)) \otimes A^{\text{op}}$, where $*_\mathbb{N}(A, \varphi)$ is the reduced free product of infinite many A 's with respect to the faithful state φ ;
3. $c_0(A) \otimes A^{\text{op}}$;

4. $C([0, 1], A) \otimes A^{\text{op}}$, where $C([0, 1], A)$ is the C^* -algebra of continuous functions on $[0, 1]$ with values in A .

As a corollary for the free product case, if we assume that the Connes embedding problem [Ki93] has a negative answer, then $C^*\mathbb{F}_\infty \otimes C^*\mathbb{F}_\infty$, not only admits different min and max norms, but also 2^{\aleph_0} distinct C^* -norms. Also for the case of the continuous functions algebra, we apply the result to the reduced group C^* -algebra and discover that if a discrete group Γ is not amenable, then there are 2^{\aleph_0} number of distinct norms on $C_\lambda^*(\mathbb{Z} \times \Gamma) \otimes C_\lambda^*\Gamma$. We conjecture that the \mathbb{Z} copies are not necessary. This conjecture would cover the result in [Wie], where Wiersma shows that if Γ contains a copy of the free group, then $C_\lambda^*\Gamma \otimes C_\lambda^*\Gamma$ admits 2^{\aleph_0} distinct norms.

By duality, we can construct norms from the quotient map $q : C^*\mathbb{F}_\infty \rightarrow A$. We show that for C^* -algebras A and B such that $A \otimes_{\min} B \neq A \otimes_{\max} B$, there are 2^{\aleph_0} distinct norms on $A \otimes B^{\otimes \mathbb{N}}$, $A \otimes (*_{\mathbb{N}}(B, \varphi))$, $A \otimes c_0(B)$, and $A \otimes C([0, 1], B)$. The construction can also be applied to the case of von Neumann algebra N with $A \otimes_{\min} N \neq A \otimes_{\text{nor}} N$.

Finally, we construct norms from the subalgebra-quotient structure. We show that if there exists a unital separable C^* -algebra A which is not QWEP, then there are 2^{\aleph_0} distinct norms on $A \otimes A^{\text{op} \otimes \mathbb{N}}$, $A \otimes (*_{\mathbb{N}}(A^{\text{op}}, \varphi))$, $A \otimes c_0(A^{\text{op}})$, and $A \otimes C([0, 1], A^{\text{op}})$.

5.1 Application to C^* -norms

In this section, we will discuss some application of our tensor norm \max^D constructed in Chapter 3, to C^* -norms. We will follow the approach in [OP14] to construct norms on $A \otimes B$ for C^* -algebras A and B .

Let E be a n -dimensional subspace in B , and $C^*\langle E \rangle$ be the separable unital C^* -subalgebra of B generated by E , which contains E completely isometrically. For free group of countably infinite generators \mathbb{F}_∞ , we have a quotient map $C^*\mathbb{F}_\infty \rightarrow B$. Let $q : C^*\langle E \rangle * C^*\mathbb{F}_\infty \rightarrow B$ denote the free product of the inclusion $C^*\langle E \rangle \subset B$ and the quotient map $C^*\mathbb{F}_\infty \rightarrow B$, and let $I = \ker(q)$, so that we have $B \simeq (C^*\langle E \rangle * C^*\mathbb{F}_\infty)/I$. Following [OP14], let

$$A \otimes_E B = \frac{A \otimes_{\min} (C^*\langle E \rangle * C^*\mathbb{F}_\infty)}{A \otimes_{\min} I}. \quad (5.1.1)$$

Similarly, we can construct a new norm using the \max_1^D norm defined in Chapter 3. Recall that for a universal inclusion $A \subset \mathcal{L}(\mathcal{H}_D^u)$, the \max_1^D norm is the induced tensor norm from the inclusion

$A \otimes C \subset \mathcal{L}(\mathcal{H}_D^u) \otimes_{\max} C$. Now we define

$$A \otimes_{D,E} B = \frac{A \otimes_{\max_1^D} (C^*\langle E \rangle * C^*\mathbb{F}_\infty)}{A \otimes_{\max_1^D} I}. \quad (5.1.2)$$

By their constructions, it is easy to see the following continuous maps

$$A \otimes_{\max} B \rightarrow A \otimes_{D,E} B \rightarrow A \otimes_E B \rightarrow A \otimes_{\min} B.$$

The goal of this section is to determine the conditions which distinguish the above norms.

We will follow the notations in [OP14]. Let us first recall the operator space duality $F^* \otimes_{\min} E \subset CB(F, E)$ isometrically (see Theorem B.13 in [BrOz]). This gives us a correspondence between a tensor $x = \sum_k f_k^* \otimes e_k \in F^* \otimes E$, and a map $\varphi_x : F \rightarrow E$ given by $\varphi_x(f) = \sum_k f_k^*(f) e_k$, with $\|x\|_{\min} = \|\varphi_x\|_{\text{cb}}$. For finite dimensional operator space E , we denote by t_E the “identity” element in $E^* \otimes E$. Note that $\|t_E\|_{\min} = 1$ and that any norm of t_E is independent of embeddings $E^* \hookrightarrow \mathbb{B}(\ell_2)$ and $E \hookrightarrow \mathbb{B}(\ell_2)$.

For any $n \in \mathbb{N}$, let \mathcal{OS}_n denote the metric space of all n -dimensional operator spaces, equipped with the completely bounded Banach-Mazur distance. Note that by [JP95], \mathcal{OS}_n is non-separable for $n \geq 3$. If A is a separable C^* -algebra, then the set $\mathcal{OS}_n(A)$ of all n -dimensional operator subspaces of A is a separable subset of \mathcal{OS}_n .

The first lemma will help us distinguish $\|\cdot\|_{E,D}$ and $\|\cdot\|_{\min}$.

Lemma 5.1.1. *Let E and F be subspaces of C^* -algebra B , and E^*, F^* be subspaces of C^* -algebra A . Then $\|t_F\|_{E,D} \geq d_{cb}(F, \mathcal{OS}_n(C^*\mathbb{F}_\infty))$, where $d_{cb}(F, \mathcal{OS}_n(C^*\mathbb{F}_\infty)) = \inf\{d_{cb}(F, G) \mid G \in \mathcal{OS}_n(C^*\mathbb{F}_\infty)\}$.*

Proof. By their construction, we have the following diagram

$$\begin{array}{ccc} A \otimes_{\max_1^D} (C^*\langle E \rangle * C^*\mathbb{F}_\infty) & \xrightarrow{\quad} & \mathcal{L}(\mathcal{H}_D^u) \otimes_{\max} (C^*\langle E \rangle * C^*\mathbb{F}_\infty) \xleftarrow{\pi} \mathcal{L}(\mathcal{H}_D^u) \otimes_{\max} C^*\mathbb{F}_\infty \\ q \downarrow & & \downarrow \iota \\ A \otimes_{E,D} B & & \mathcal{L}(\mathcal{H}_D^u) \otimes_{\min} C^*\mathbb{F}_\infty \end{array}$$

where π is induced from a quotient map $C^*\mathbb{F}_\infty \rightarrow C^*\langle E \rangle * C^*\mathbb{F}_\infty$, and ι is a continuous inclusion.

Note that for finite dimensional $F^* \subset A$ and $F \subset B$, we can lift F to a subspace $G \subset C^*\langle E \rangle * C^*\mathbb{F}_\infty$, and then a $\tilde{G} \subset C^*\mathbb{F}_\infty$. Therefore the identity map t_F on $F^* \otimes F \subset A \otimes_{E,D} B$ admits a lifting

$\xi \in F^* \otimes \tilde{G} \subset \mathcal{L}(\mathcal{H}_D^u) \otimes_{\min} C^*\mathbb{F}_\infty$ which corresponds to a map $\alpha : F \rightarrow \tilde{G}$. Hence we have a factorization

$$F \xrightarrow{\alpha} \tilde{G} \xrightarrow{q \circ \pi} B,$$

such that the composition is the inclusion $F \subset B$. Therefore the image $\alpha(F)$ in \tilde{G} is isomorphic to F . Hence we have

$$\begin{array}{ccccc} t_F : F & \longrightarrow & \alpha(F) & \longrightarrow & F \\ & & \downarrow & & \\ & & C^*\mathbb{F}_\infty & & \end{array}$$

and therefore, $\|t_F\|_{E,D} \geq d_{cb}(F, \mathcal{OS}_n(C^*\mathbb{F}_\infty))$. \square

The next lemma will help us distinguish $\|\cdot\|_{E,D}$ and $\|\cdot\|_{\max}$.

Lemma 5.1.2. *Let $A \subset \mathcal{L}(\mathcal{H}_D^u)$ be the universal representation of A . Also let π be a surjective u.c.p. from B_1 to B with kernel I , such that $(A \otimes_{\max_1^D} B_1)/(A \otimes_{\max_1^D} I) \simeq A \otimes_{\max} B$. If there exists a surjective completely positive map $\sigma : B \rightarrow A^{\text{op}}$, then A has the DWEP₁.*

Proof. From the assumptions, we have the following diagram

$$\begin{array}{ccccccc} A \otimes_{\max_1^D} B_1 & \longrightarrow & A \otimes_{\max} B & \longrightarrow & A \otimes_{\max} A^{\text{op}} & \longrightarrow & \mathbb{B}(L_2(A^{**})) \\ \downarrow & & & & & \nearrow & \\ \mathcal{L}(\mathcal{H}_D^u) \otimes_{\max} B_1 & & & & & & \end{array}$$

By Arveson's extension theorem, there exists a u.c.p. map $\Phi : \mathcal{L}(\mathcal{H}_D^u) \otimes_{\max} B_1 \rightarrow \mathbb{B}(L_2(A^{**}))$. Applying *the Trick*, we obtain a u.c.p. map $\phi : \mathcal{L}(\mathcal{H}_D^u) \rightarrow A^{**}$, which is identity on A , and hence A has the DWEP₁. \square

Now we are ready to give the conditions which distinguish the norms.

Theorem 5.1.3. *Consider the four norms on $A \otimes B$*

$$A \otimes_{\min} B \neq A \otimes_{(a)} B \neq A \otimes_E B \neq_{(b)} A \otimes_{D,E} B \neq_{(c)} A \otimes_{\max} B.$$

The strict inequality holds for

(a), *if there exists n -dimensional subspaces $F^* \subset A$ and $F \subset B$, such that $F \notin \mathcal{OS}_n(C^*\langle E \rangle * C^*\mathbb{F}_\infty)$;*

(b), if $E \notin \mathcal{OS}_n(C^*\mathbb{F}_\infty)$;

(c), if there exists a surjective u.c.p. $B \rightarrow A^{\text{op}}$, and A does not have the $DWEP_1$.

Moreover, for n -dimensional subspaces E, F in B , and E^*, F^* in A , we have $A \otimes_{D,E} B \neq A \otimes_F B$, if $F \notin \mathcal{OS}_n(C^*\mathbb{F}_\infty)$. Therefore $A \otimes_{D,E} B$ gives us a new norm on $A \otimes B$, distinct from the continuum norms constructed in [OP14].

Proof. (a) is proved in [OP14]. Indeed, if such F and F^* exist, then the identity map t_F on $F^* \otimes F \subset A \otimes_{\min} B$ has norm 1. On the other hand, notice that the norm of t_F in $A \otimes_E B$ is greater than 1. Indeed if $\|t_F\|_E = 1$, then by the construction of $A \otimes_E B$, it lifts to an element $\xi \in F^* \otimes (C^*\langle E \rangle * C^*\mathbb{F}_\infty)$ with $\|\xi\|_{\min} = 1$. This corresponds to a completely isometric mapping $F \rightarrow C^*\langle E \rangle * C^*\mathbb{F}_\infty$, showing that F is completely isometric to a subspace of $C^*\langle E \rangle * C^*\mathbb{F}_\infty$, which contradicts the condition $F \notin \mathcal{OS}_n(C^*\langle E \rangle * C^*\mathbb{F}_\infty)$. Hence $\|t_F\|_E > \|t_F\|_{\min} = 1$.

(b) By Lemma 5.1.1, $\|t_E\|_{E,D} \geq d_{\text{cb}}(E, \mathcal{OS}_n(C^*\mathbb{F}_\infty))$. If $E \notin \mathcal{OS}_n(C^*\mathbb{F}_\infty)$, then we have $d_{\text{cb}}(E, \mathcal{OS}_n(C^*\mathbb{F}_\infty)) > 1$, and so is $\|t_E\|_{E,D}$. Therefore $\|t_E\|_{E,D} > \|t_E\|_E = 1$.

(c) Apply Lemma 5.1.2 to $B_1 = C^*\langle E \rangle * C^*\mathbb{F}_\infty$. Then by the construction we have $A \otimes_{D,E} B = (A \otimes_{\max_I^P} B_1) / (A \otimes_{\max_I^P} I)$. If $A \otimes_{D,E} B = A \otimes_{\max} B$, then by Lemma 5.1.2, A has the $DWEP_1$, which contradicts the condition.

Moreover, similar to the proof of (b), Lemma 5.1.1 shows that $A \otimes_{D,E} B \neq A \otimes_F B$, if $F \notin \mathcal{OS}_n(C^*\mathbb{F}_\infty)$. \square

Now we will construct C^* -algebras A and B giving a concrete example with the above distinct norms. Our goal is to construct a C^* -algebra A such that $A \simeq A^{\text{op}}$ without $DWEP_1$, and let $B = A$.

Recall that for operator spaces E and F , $C^*\langle E \rangle * C^*\langle F \rangle \simeq C^*\langle E \otimes_h F \rangle$, where $E \otimes_h F$ is the Haagerup tensor product, and also that $C^*\langle E^{\text{op}} \rangle \simeq C^*\langle E \rangle^{\text{op}}$.

Lemma 5.1.4. *Let $C = C^*\langle E \otimes_h E^{\text{op}} \rangle$. Then $C \simeq C^{\text{op}}$.*

Proof. Let $\pi : C^*\mathbb{F}_\infty \rightarrow C^*\langle E \rangle$ be the quotient map, then so is $\pi^{\text{op}} : C^*\mathbb{F}_\infty^{\text{op}} \rightarrow C^*\langle E \rangle^{\text{op}}$. Then we have a quotient map $C^*\mathbb{F}_\infty * C^*\mathbb{F}_\infty^{\text{op}} \rightarrow C^*\langle E \rangle * C^*\langle E \rangle^{\text{op}}$, which maps the unitaries to unitaries. Notice

that for I the index set of \mathbb{F}_∞ , we have the following isomorphism given by

$$\begin{array}{ccccccc}
C^*\mathbb{F}_\infty * C^*\mathbb{F}_\infty^{\text{op}} & \simeq & C^*\mathbb{F}_{I \times I} & \simeq & (C^*\mathbb{F}_{I \times I})^{\text{op}} \\
g_i * 1 & \longmapsto & 1 \times g_i^{-1} & \longmapsto & (1 \times g_i)^{\text{op}} \\
1 * h_i & \longmapsto & h_i^{-1} \times 1 & \longmapsto & (h_i \times 1)^{\text{op}}
\end{array}$$

Let $\pi(g_i) = x$ and $\pi^{\text{op}}(h_i) = y^{\text{op}}$. Define the map $C^*\langle E \rangle * C^*\langle E \rangle^{\text{op}} \rightarrow (C^*\langle E \rangle * C^*\langle E \rangle^{\text{op}})^{\text{op}}$, by $x * 1 \rightarrow (1 * x^{\text{op}})^{\text{op}}$, and $1 * y^{\text{op}} \rightarrow (y * 1)^{\text{op}}$. Then it is easy to check that this is an isomorphism following from the isomorphism $C^*\mathbb{F}_\infty * C^*\mathbb{F}_\infty^{\text{op}} \rightarrow (C^*\mathbb{F}_{I \times I})^{\text{op}}$. \square

Now we are ready to construct the example. For n -dimensional operator spaces E and F satisfying the conditions (a) and (b) in Theorem 5.1.3, let

$$D = C^*\langle (E \oplus E^* \oplus F \oplus F^*) \otimes_h (E \oplus E^* \oplus F \oplus F^*)^{\text{op}} \rangle,$$

where the direct sum is in ℓ_∞ . Then by Lemma 5.1.4, $D \simeq D^{\text{op}}$.

Let $A = D \otimes_{\min} C_\lambda^*\mathbb{F}_2$. Then we have

$$A^{\text{op}} = (D \otimes_{\min} C_\lambda^*\mathbb{F}_2)^{\text{op}} \simeq D^{\text{op}} \otimes_{\min} C_\lambda^*\mathbb{F}_2^{\text{op}} \simeq D \otimes_{\min} C_\lambda^*\mathbb{F}_2 = A.$$

Let $B = A$, and hence we have a surjective u.c.p. $B \rightarrow A^{\text{op}}$. Also since $C_\lambda^*\mathbb{F}_2$ does not have the WEP, the faithful representation for $C_\lambda^*\mathbb{F}_2 \subset \mathbb{B}(\mathcal{H})$ induces an inclusion $A = D \otimes_{\min} C_\lambda^*\mathbb{F}_2 \hookrightarrow D \otimes_{\min} \mathbb{B}(\mathcal{H})$ which is not *r.w.i.*. Notice that this is not equivalent to DWEP. However if we construct the \max^D norm from the inclusion $A \subseteq D \otimes_{\min} \mathbb{B}(\mathcal{H})$ instead of $A \subseteq \mathcal{L}(\mathcal{H}_D^u)$, we will have the same conclusion that the four norms are distinct.

Corollary 5.1.5. *Let D be as above, and $A = D \otimes_{\min} C_\lambda^*\mathbb{F}_2$. For a faithful representation $C_\lambda^*\mathbb{F}_2 \subset \mathbb{B}(\mathcal{H})$, define the \max^D norm on $A \otimes C$ to be the tensor norm induced from the inclusion $A \otimes C \subseteq (D \otimes_{\min} \mathbb{B}(\mathcal{H})) \otimes_{\max} C$. Let $B = A$. Define the quotient norms $A \otimes_E B$ as in (5.1.1) and $A \otimes_{D,E} B$ as in (5.1.2) with the new \max^D norm as follows*

$$A \otimes_{D,E} B = \frac{A \otimes_{\max^D} (C^*\langle E \rangle * C^*\mathbb{F}_\infty)}{A \otimes_{\max^D} I}.$$

Then we have the following strict norms

$$A \underset{\min}{\otimes} B \neq A \underset{E}{\otimes} B \neq A \underset{D,E}{\otimes} B \neq A \underset{\max}{\otimes} B.$$

5.2 Construction of norms from subalgebras

Lemma 5.2.1. *Let A and B be separable C^* -algebras. Then there are at most 2^{\aleph_0} number of distinct norms on $A \otimes B$.*

Proof. For an arbitrary tensor norm $\|\cdot\|_\alpha$ on $A \otimes B$, by the universal property of the max-tensor product, there exists a $*$ -homomorphism $q_\alpha : A \otimes_{\max} B \rightarrow A \otimes_\alpha B$. Let $X \subset A \otimes B$ be a countable dense subset with respect to the max-tensor norm. For tensor norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ on $A \otimes B$, $q_\alpha(x) = q_\beta(x)$ for any $x \in X$ implies that $\|\cdot\|_\alpha = \|\cdot\|_\beta$. This gives rise to an injective map from the set of tensor norms to the set of functions from X to \mathbb{R} . Notice that the later has the cardinality of continuum, and hence we have the desired result. \square

From now on, C^* -algebras are assumed to be unital and separable throughout the paper.

Theorem 5.2.2. *Suppose that A is a unital subalgebra of C^* -algebra B , which is not r.w.i. in B . Let $A^{\otimes \mathbb{N}}$ be the tensor product $A \otimes A \cdots \otimes A$ of countable copies of A , completed in $\|\cdot\|_{\min}$ norm. Then there are 2^{\aleph_0} number of distinct norms on $A^{\otimes \mathbb{N}} \otimes A^{\text{op}}$.*

Proof. It is clear that for $j \in J$, there is a natural $*$ -homomorphism $\pi_j : A \rightarrow A^{\otimes \mathbb{N}}$, defined as $\pi_j(a) = 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$ on the j -th component of $A^{\otimes \mathbb{N}}$.

For each subset $J \subset \mathbb{N}$, define the algebra $D_J = D_J^{\otimes \mathbb{N}}$, closed in the min-norm, where

$$D_J = \begin{cases} A & \text{if } j \in J; \\ B & \text{if } j \notin J. \end{cases} \quad (5.2.1)$$

Then for the inclusion $\iota : A \hookrightarrow B$ and a faithful state φ on B , we can define the $*$ -homomorphism $\Theta_J = \otimes \theta_j : A^{\otimes \mathbb{N}} \rightarrow D_J$ by

$$\theta_j = \begin{cases} \text{id}_A & \text{if } j \in J; \\ \iota & \text{if } j \notin J. \end{cases}$$

and $\varphi_j = \varphi \otimes \cdots \otimes \text{id} \otimes \cdots \otimes \varphi$, where the identity is on the j -th component.

Now for each $J \subset \mathbb{N}$, we define the (J, \max) -norm to be the closure of $(\Theta_J \otimes \text{id})(A^{\otimes \mathbb{N}} \otimes A^{\text{op}})$ in $D_J \otimes_{\max} A^{\text{op}}$ with respect to the max-tensor norm, and we regard $A^{\otimes \mathbb{N}} \otimes_{J, \max} A^{\text{op}}$ as a C^* -subalgebra in $D_J \otimes_{\max} A^{\text{op}}$. We claim that for different subset J and J' , the norms are different, and hence we have 2^{\aleph_0} number of distinct norms on $A^{\otimes \mathbb{N}} \otimes A^{\text{op}}$.

Without loss of generality, let $j \in J \setminus J'$, we have maps

$$A \otimes A^{\text{op}} \xrightarrow{\pi_j \otimes \text{id}} A^{\otimes \mathbb{N}} \otimes_{J, \max} A^{\text{op}} \xrightarrow{\Theta_J \otimes \text{id}} D_J \otimes_{\max} A^{\text{op}} \xrightarrow{\varphi_j \otimes \text{id}} A \otimes_{\max} A^{\text{op}} \rightarrow \mathbb{B}(\mathcal{H}_u),$$

where \mathcal{H}_u is the universal representation of A .

Suppose that $A^{\otimes \mathbb{N}} \otimes_{J, \max} A^{\text{op}} = A^{\otimes \mathbb{N}} \otimes_{J', \max} A^{\text{op}}$. We can apply Arveson's extension theorem to $A^{\otimes \mathbb{N}} \otimes_{J', \max} A^{\text{op}} \subset D_{J'} \otimes_{\max} A^{\text{op}}$. This yields a u.c.p. map $\Phi : D_J \otimes_{\max} A^{\text{op}} \rightarrow \mathbb{B}(\mathcal{H})$. Notice that $\Phi|_{\pi_j(B) \otimes 1} = \varphi_j \otimes \text{id}|_{\pi_j(A) \otimes 1} \subset A^{**}$. Define a map $T : B \rightarrow A^{**}$ by $T(b) = \Phi \circ (\pi_j(b) \otimes 1)$ for $b \in B$. It is easy to see that it preserves the identity on A , which contradicts the assumption that A is not *r.w.i.* in B . \square

Remark 5.2.3. The core of the above proof is referred as *The Trick* by Kirchberg [Ki93] (see more details in [BrOz] Prop. 3.6.5). If we define the (\max, B) -norm on $A \otimes C$ such that the inclusion $A \otimes_{\max, B} C \subset A \otimes_{\max} B$ is injective, then the *r.w.i.* property of the inclusion $A \subset B$ is equivalent to say $A \otimes_{\max, B} C = A \otimes_{\max} C$ for any C^* -algebra C . Therefore for $j \in J$ and $j' \in J'$, we have that

$$A \otimes_{\max, B} A^{\text{op}} \subset B \otimes_{\max} A^{\text{op}} \xrightarrow{\pi_j \otimes \text{id}} D_J \otimes_{\max} A^{\text{op}} \xrightarrow{\varphi_{j'} \otimes \text{id}} A \otimes_{\max} A^{\text{op}},$$

which contradicts the assumption that A is not *r.w.i.* in B .

Our method can be applied more generally. Fix a unital inclusion $A \subset B$. Assume \tilde{A} is a C^* -algebra constructed from countably infinite copies of A , such that for $j \in \mathbb{N}$, there exists a component map $\pi_j : A \rightarrow \tilde{A}$. Accordingly suppose we have \tilde{B} and $\pi_j : B \rightarrow \tilde{B}$, such that the induced map from \tilde{A} to \tilde{B} is injective and the following diagram commutes:

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \pi_j \downarrow & & \downarrow \pi_j \\ \tilde{A} & \hookrightarrow & \tilde{B} \end{array}$$

For a subset $J \subset \mathbb{N}$, define D_J componentwise by D_J , compatible with the construction of \tilde{A} from

A and the component map π_j , such that $D_j = A$ if $j \in J$, and $D_j = B$ if $j \notin J$, i.e., the following diagram commutes:

$$\begin{array}{ccccc} A & \longrightarrow & D_j & \longrightarrow & B \\ \downarrow \pi_j & & \downarrow \pi_j & & \downarrow \pi_j \\ \tilde{A} & \hookrightarrow & D_J & \hookrightarrow & \tilde{B} \end{array}$$

Moreover, we have a c.p. subunital map $\varphi_j : D_J \rightarrow D_j$ such that $\varphi_j \circ \pi_j|_{D_j} = \text{id}_{D_j}$.

As an illustration of such conditions, the min-tensor product $A^{\otimes \mathbb{N}}$ and $B^{\otimes \mathbb{N}}$ can play the role of \tilde{A} and \tilde{B} respectively. Define $D_J = D_j^{\otimes \mathbb{N}}$, where D_j are as in (5.2.1). Hence we have $A^{\otimes \mathbb{N}} \subset D_J \subset B^{\otimes \mathbb{N}}$, with component maps π_j for each $j \in \mathbb{N}$. Another example is the reduced free product with respect to a faithful state φ on B . By [VDN] we have injective inclusions $*_{\mathbb{N}}(A, \varphi) \subset *_{j \in \mathbb{N}}(D_j, \varphi) \subset *_{\mathbb{N}}(B, \varphi)$.

For a C^* -algebra C , we can define the J -norm on $\tilde{A} \otimes C$, which is closed in max-norm as a subalgebra in $D_J \otimes_{\max} C$, namely

$$\tilde{A} \otimes_J C \subset D_J \otimes_{\max} C.$$

Assume that the inclusion $A \subset B$ is not *r.w.i.* and C admits A^{op} as a quotient, i.e. there is a u.c.p. map from C onto A^{op} . Then for each different subset J and J' , we have different J and J' -norms on $\tilde{A} \otimes C$ by Remark 5.2.3.

Now we can apply our construction in the following situations.

Proposition 5.2.4. *Suppose that A is a unital subalgebra of C^* -algebra B , which is not *r.w.i.* in B , and C is a C^* -algebra which admits A^{op} as a quotient. Then there are 2^{\aleph_0} distinct norms on the following C^* -algebras:*

1. $*_{\mathbb{N}}(A, \varphi) \otimes C$, where φ is a faithful state on B , and $*_{\mathbb{N}}(A, \varphi)$ is the reduced free product of countable many A 's associated with φ .
2. $c_0(A) \otimes C$.
3. $C([0, 1], A) \otimes C$, where $C([0, 1], A)$ is the C^* -algebra of continuous functions on interval $[0, 1]$ with values in A .

Proof. (1). For $J \subset \mathbb{N}$, let $D_J = *_{j \in \mathbb{N}}(D_j, \varphi)$, where $D_j = A$ if $j \in J$, and $D_j = B$ if $j \notin J$. By [VDN], we have injective inclusions $*_{\mathbb{N}}(A, \varphi) \subset *_{j \in \mathbb{N}}(D_j, \varphi) \subset *_{\mathbb{N}}(B, \varphi)$.

Now define the J -norm on $*_{\mathbb{N}}(A, \varphi) \otimes C$ from the inclusion of algebraic tensor product such that

$$(*_{\mathbb{N}}(A, \varphi)) \otimes_J C \subset (*_{j \in \mathbb{N}}(D_j, \varphi)) \otimes_{\max} C.$$

This allows us to track the j -th position as in Theorem 5.2.2 and hence we find continuum number of distinct norms.

(2). We view that $c_0(A)$ as the c_0 -direct sum of A . For $J \in \mathbb{N}$, let $D_J = \oplus_{j \in \mathbb{N}} D_j$, where $D_j = A$ if $j \in J$, and $D_j = B$ if $j \notin J$. Define the J -norm on $c_0(A) \otimes C$ from the inclusion of algebraic tensor product such that

$$c_0(A) \otimes_J C \subset (\oplus D_j) \otimes_{\max} C,$$

where $D_j = A$ if $j \in J$, and $D_j = B$ if $j \notin J$. Moreover, we have the component maps π_j , and conditional expectations $\varphi_j : *_j \in \mathbb{N} (D_j, \varphi) \rightarrow (D_j, \varphi)$. By tracking the j -th component, we have the desired result.

(3). Let $0 \leq t_1 < t_2 < \dots < t_j < t_{j+1} < \dots < 1$ be a sequence in $[0, 1]$, and $s_j \in (t_j, t_{j+1})$ be the midpoint of the interval. For $J \subset \mathbb{N}$, define the algebra D_J as

$$D_J = \{f : [0, 1] \rightarrow B \mid f(t_j) \in A, \forall j \in \mathbb{N}; f(s) \in A, \forall j \in J \text{ and } t_j < s < t_{j+1}\}.$$

Then we have an evaluation map ev_1 such that

$$\begin{aligned} D_J &\xrightarrow{\text{ev}_1} \ell_\infty(B) \\ f &\mapsto (f(s_j))_{j \in \mathbb{N}} \end{aligned}$$

Now let $\tilde{D}_J = \Pi_{j=1}^\infty D_j$, with $D_j = A$ for $j \in J$, and $D_j = B$ for $j \notin J$. The map ev_1 admits a c.p. lifting v such that

$$\begin{aligned} \tilde{D}_J &\xrightarrow{v} D_J \\ (x_j)_{j \in \mathbb{N}} &\mapsto \sum_{j \in \mathbb{N}} x_j g_j \end{aligned}$$

where $g_j \in C[0, 1]$ with $\text{supp } g_j \subset (t_j, t_{j+1})$, $g_j(s_j) = 1$, and $\sum_{j \in \mathbb{N}} g_j(s) = 1$. It is easy to verify that D_J is a C^* -algebra.

Now we define the J -norm as

$$C([0, 1], A) \otimes_J C \subset D_J \otimes_{\max} C.$$

Therefore we have component maps $\pi_j : A \otimes C \rightarrow C([0, 1], A) \otimes_J C$ such that $\pi_j(a \otimes c) = g_j a \otimes c$, and

component conditional expectation $\varphi_j : C([0, 1], A) \otimes_J C \rightarrow A \otimes_{\max C}$ as $\varphi_j(f \otimes c) = f(s_j)c$. The rest of the proof follows as in Theorem 5.2.2. \square

As an application of the first statement in Proposition 5.2.4, we consider the tensor norms on $C^*\mathbb{F}_\infty \otimes C^*\mathbb{F}_\infty$. The QWEP conjecture [Ki93] says that all C^* -algebras are QWEP, which is equivalent to the fact that $C^*\mathbb{F}_\infty$ has the WEP, or equivalently $C^*\mathbb{F}_\infty \otimes_{\min} C^*\mathbb{F}_\infty = C^*\mathbb{F}_\infty \otimes_{\max} C^*\mathbb{F}_\infty$. Now if the QWEP conjecture were not true, then we would have not only two, but a continuum number of distinct norms on $C^*\mathbb{F}_\infty \otimes C^*\mathbb{F}_\infty$.

Corollary 5.2.5. *If the QWEP conjecture has a negative answer, then there are 2^{\aleph_0} number of distinct norms on $C^*\mathbb{F}_\infty \otimes C^*\mathbb{F}_\infty$.*

Proof. We have $C^*\mathbb{F}_\infty \simeq C^*\mathbb{F}_{\mathbb{N} \times \mathbb{N}} \simeq {}^*\mathbb{N}C^*\mathbb{F}_\mathbb{N}$. If the QWEP conjecture has a negative answer, then $C^*\mathbb{F}_\mathbb{N}$ is not *r.w.i.* in $\mathbb{B}(\mathcal{H})$. By Proposition 5.2.4, we can construct continuum number of distinct norms on $C^*\mathbb{F}_\mathbb{N} \otimes C^*\mathbb{F}_\mathbb{N}$. \square

For an application of the second statement, we consider the case of group C^* -algebras. Recall that for a discrete group Γ , the reduced group C^* -algebra $C_\lambda^*\Gamma$ has the WEP if and only if Γ is amenable (see [BrOz], Prop. 3.6.9.).

Corollary 5.2.6. *Suppose Γ is a non-amenable discrete group. Then $C_\lambda^*(\Gamma \times \mathbb{Z}) \otimes C_\lambda^*\Gamma$ admits 2^{\aleph_0} number of distinct norms.*

Proof. Notice that $C_\lambda^*(\Gamma \times \mathbb{Z}) \simeq C(\mathbb{T}, C_\lambda^*\Gamma)$. Then Statement (2) in Corollary 5.2.4 yields the assertion. \square

The Corollary 5.2.6 is related to Wiersma's result in [Wie] which finds a continuum number of norms on $C_\lambda^*\Gamma_1 \otimes C_\lambda^*\Gamma_2$, if Γ_1 and Γ_2 contain some free group. We conjecture that the \mathbb{Z} -copies in the Corollary 5.2.6 are not necessary, namely $C_\lambda^*\Gamma \otimes C_\lambda^*\Gamma$ admits 2^{\aleph_0} different norms, whenever Γ is not amenable. This would cover Wiersma's result.

5.3 Constructions of norms from quotients

In the previous section, the main idea is to use the non-relatively weak injectivity of an inclusion $A \subset B$, and construct normed copies either from the max norm or from the max-induced norms on subalgebras. In this section, we will construct the norms from quotients.

Theorem 5.3.1. *For C^* -algebras A and B , such that $A \otimes_{\min} B \neq A \otimes_{\max} B$, there are a continuum number of distinct norms on $A \otimes B^{\otimes \mathbb{N}}$, where $B^{\otimes \mathbb{N}}$ is the tensor product of countably infinite copies of B completed in the min-norm.*

In order to construct a norm for each subset $J \subset \mathbb{N}$, we first consider the quotient map. For any unital C^* -algebra A , there exists a quotient map $q : C^*\mathbb{F}_\infty \rightarrow A$. This quotient map q induces u.c.p. maps $Q^{\max} : C^*\mathbb{F}_\infty \otimes_{\max} B \rightarrow A \otimes_{\max} B$ and $Q^{\min} : C^*\mathbb{F}_\infty \otimes_{\max} B \rightarrow A \otimes_{\min} B$. Let I^{\max} and I^{\min} be the kernels of Q^{\max} and Q^{\min} respectively. For each $i \in \mathbb{N}$, we have a natural $*$ -homomorphism $\pi_i : B \rightarrow B^{\otimes \mathbb{N}}$, such that $\pi_i(b) = 1 \otimes 1 \otimes \cdots \otimes b \otimes \cdots \otimes 1$ on the i -th component. Now for each subset $J \subset \mathbb{N}$, let

$$I_J = \begin{cases} I^{\max} & \text{if } j \in J; \\ I^{\min} & \text{if } j \notin J. \end{cases} \quad (5.3.1)$$

and let I_J be the ideal in $C^*\mathbb{F}_\infty \otimes_{\max} B^{\otimes \mathbb{N}}$ generated by $1 \otimes \pi_j(I_j)$ for all $j \in \mathbb{N}$. Now we define the J -norm on $A \otimes B^{\otimes \mathbb{N}}$ to be the induced quotient norm as

$$A \otimes_J B^{\otimes \mathbb{N}} = \frac{C^*\mathbb{F}_\infty \otimes_{\max} B^{\otimes \mathbb{N}}}{I_J}.$$

We will show that each different subset $J \subset \mathbb{N}$ gives rise to a distinct J -norm.

Lemma 5.3.2. *Let φ be a faithful state on B and E_j be the conditional expectation from $C^*\mathbb{F}_\infty \otimes_{\max} B^{\otimes \mathbb{N}}$ to $C^*\mathbb{F}_\infty \otimes_{\max} B$, such that $E_j = \text{id}_{C^*\mathbb{F}_\infty} \otimes \varphi \otimes \cdots \otimes \text{id}_B \otimes \cdots \otimes \varphi$, where id_B is on the j -th position in $B^{\otimes \mathbb{N}}$. Then we have the following:*

- (a) *For $j \in J$, the conditional expectation E_j on $C^*\mathbb{F}_\infty \otimes_{\max} B^{\otimes \mathbb{N}}$ induces a well-defined u.c.p. map \hat{E}_j on $A \otimes_J B^{\otimes \mathbb{N}}$ such that the following diagram commutes:*

$$\begin{array}{ccc} C^*\mathbb{F}_\infty \otimes_{\max} B^{\otimes \mathbb{N}} & \xrightarrow{E_j} & C^*\mathbb{F}_\infty \otimes_{\max} B \\ \downarrow & & \downarrow Q^{\max} \\ A \otimes_J B^{\otimes \mathbb{N}} & \xrightarrow{\hat{E}_j} & A \otimes_{\max} B \end{array}$$

- (b) *The map $\text{id}_{C^*\mathbb{F}_\infty} \otimes \pi_j$ on $C^*\mathbb{F}_\infty \otimes_{\max} B$ induces a continuous map $\hat{\pi}_j$ such that for $j \in J$, the following composition is the identity:*

$$A \otimes_{\max} B \xrightarrow{\hat{\pi}_j} A \otimes_J B^{\otimes \mathbb{N}} \xrightarrow{\hat{E}_j} A \otimes_{\max} B;$$

and for $j \notin J$, the following composition is the identity:

$$A \otimes_{\min} B \xrightarrow{\hat{\pi}_j} A \otimes_J B^{\otimes \mathbb{N}} \xrightarrow{\hat{E}_j} A \otimes_{\min} B.$$

Proof. (a) Notice that $\ker Q^{\max} \subset \ker Q^{\min}$, and hence it suffices to show that \hat{E}_j is well defined from $A \otimes_J B^{\otimes \mathbb{N}}$ to $A \otimes_{\max} B$, and the digram commutes for Q^{\max} . We claim that $I_J \subset \ker(Q^{\max} \circ E_j)$, and therefore \hat{E}_j is well defined.

Indeed, we have the commuting diagram

$$\begin{array}{ccc} C^*\mathbb{F}_\infty \otimes_{\max} B & \xrightarrow{\text{id} \otimes \varphi} & C^*\mathbb{F}_\infty \\ \downarrow & & \downarrow q \\ A \otimes_{\min} B & \xrightarrow{\text{id} \otimes \varphi} & A \end{array}$$

Hence for $z \in I^{\min}$, $\text{id} \otimes \varphi(z)$ might not be 0, but we have

$$\inf_{\alpha \in \ker q} \|\alpha \cdot (\text{id} \otimes \varphi)(z)\| = 0. \quad (5.3.2)$$

Let \hat{I}_i be the ideal generated by $\pi_i(I_i)$. This means $\hat{I}_i = (C^*\mathbb{F}_\infty \otimes_{\max} B^{\otimes \mathbb{N}}) \cdot (\text{id} \otimes \pi_i)(I_i) \cdot (C^*\mathbb{F}_\infty \otimes_{\max} B^{\otimes \mathbb{N}})$, and $I_J = \sum_{i \in \mathbb{N}} \hat{I}_i$. Let $z = \sum_k a_k \otimes b_k \in I_i$, and $T = 1_{C^*\mathbb{F}_\infty} \otimes T_1 \otimes \cdots \otimes T_j \otimes \cdots \in C^*\mathbb{F}_\infty \otimes_{\max} B^{\otimes \mathbb{N}}$. Then elements in I_J can be written as combinations of $z_j = \sum_k a_k \otimes T_1 \otimes \cdots \otimes T_{j-1} \otimes b_k \otimes T_{j+1} \otimes \cdots$.

For $i \neq j$, let $Z_j = E_j(z_i)$. Then $Z_j = \sum_k a_k \varphi(b_k) \Pi_{r \neq j} \varphi(T_r) T_j$, and therefore by (5.3.2), there exists an α such that $\|(\alpha \otimes \text{id})Z_j\| < \varepsilon$. Notice that $\ker Q^{\max} = \ker q \otimes_{\max} B$, and hence we have $\alpha \otimes \text{id} \in \ker Q^{\max}$. This implies that $Q^{\max}((\alpha \otimes \text{id})Z_j) = Q^{\max}(Z_j)$. Therefore $\|Q^{\max}(Z_j)\| = \|Q^{\max}((\alpha \otimes \text{id})Z_j)\| < \varepsilon$.

For $i = j$, we have $Z_j = \sum_k a_k \otimes b_k \Pi_{r \neq j} \varphi(T_r)$, and hence $Q^{\max}(Z_j) = 0$.

(b) We will only show the max-case. The min-case is similar. Let $\gamma \in \ker Q^{\max} \subset C^*\mathbb{F}_\infty \otimes_{\max} B$. Since I_J is the kernel of the quotient map from $C^*\mathbb{F}_\infty \otimes_{\max} B^{\otimes \mathbb{N}}$ to $A \otimes_J B^{\otimes \mathbb{N}}$, we have that $(\text{id} \otimes \pi_j)(\gamma) \in \hat{I}_j \subset I_J$. Therefore the induced map on $A \otimes_{\max} B$ is well defined and the following diagram commutes:

$$\begin{array}{ccc} C^*\mathbb{F}_\infty \otimes_{\max} B & \xrightarrow{\text{id} \otimes \pi_j} & C^*\mathbb{F}_\infty \otimes_{\max} B^{\otimes \mathbb{N}} \\ \downarrow & & \downarrow \\ A \otimes_{\max} B & \xrightarrow{\hat{\pi}_j} & A \otimes_J B^{\otimes \mathbb{N}} \end{array}$$

In order to the identity composition, let $\eta = \sum_k a_k \otimes b_k \in A \otimes_m B$ with $\|\eta\|_{\max} < 1$. Then there exists $\xi \in C^*\mathbb{F}_\infty$ such that $Q^{\max}(\xi) = \eta$. From (a), we have that $E_j \circ (\text{id} \otimes \pi_j)(\xi) = \xi$, and hence $Q^{\max} \circ E_j \circ (\text{id} \otimes \pi_j)(\xi) = \eta$. Since the diagram commutes, we have $\hat{E}_j \circ \hat{\pi}_j(\eta) = \eta$. \square

Now we are ready to prove the main theorem of this section.

Proof of Theorem 5.3.1. For different subset J and J' in \mathbb{N} . Let $j \in J' \setminus J$. By (b) in Lemma 5.3.2, we have

$$A \otimes_{\min} B \xrightarrow{\hat{\pi}_j} A \otimes_J B^{\otimes \mathbb{N}} = A \otimes_{J'} B^{\otimes \mathbb{N}} \xrightarrow{\hat{E}_j} A \otimes_{\max} B, \quad (5.3.3)$$

which contradicts the assumption that $A \otimes_{\min} B \neq A \otimes_{\max} B$. \square

Similarly as in the previous section, the construction on infinite tensors can be modified to the other cases of infinite copies such as the reduced free product, the c_0 -direct sum and the continuous B -valued function algebra.

Proposition 5.3.3. *For C^* -algebras A and B , such that $A \otimes_{\min} B \neq A \otimes_{\max} B$, there are a continuum number of distinct norms on the following algebras:*

1. $A \otimes *_{\mathbb{N}}(B, \varphi)$;
2. $A \otimes c_0(B)$;
3. $A \otimes C([0, 1], B)$.

Proof. (1) follows exactly as the construction of J -norms in the case of infinite tensor product.

(2) The construction is even easier. Since direct sum and tensor product commute, we can define the J -norm on $A \otimes c_0(B)$ from the inclusion $A \otimes c_0(B) \subset \oplus_{j \in \mathbb{N}} (A \otimes_{m_j} B)$, where the m_j -norm is the min-norm if $j \in J$, and the max-norm if $j \notin J$. Hence we have well-defined component maps $\hat{\pi}_j$ and conditional expectations \hat{E}_j . The result follows as in the proof of the Theorem 5.3.1.

(3) Let $(t_i)_{i \in \mathbb{N}}$ be an increasing sequence in $[0, 1]$, and s_j be the midpoint for interval (t_j, t_{j+1}) . For $J \subset \mathbb{N}$, let I_J as in (5.3.1), and define

$$I_J = \{f : [0, 1] \rightarrow C^*\mathbb{F}_\infty \otimes_{\max} B \mid f(s) \in I_J \text{ for } t_j < s < t_{j+1} \text{ and } j \in J; f(t_i) = 0 \text{ for all } i \in \mathbb{N}\}$$

Notice that $C^*\mathbb{F}_\infty \otimes_{\max} C([0, 1], B) \simeq C^*\mathbb{F}_\infty \otimes_{\max} B \otimes_{\max} C[0, 1] \simeq C([0, 1], C^*\mathbb{F}_\infty \otimes_{\max} B)$. Therefore we have $I_J = \sum_{i \in \mathbb{N}} I_i$ and in particular I_J is the ideal generated by all the I_i 's in $C^*\mathbb{F}_\infty \otimes_{\max} C([0, 1], B)$.

Now define the J -norm on $A \otimes C([0, 1], B)$ such that

$$A \otimes_J C([0, 1], B) = \frac{C^*\mathbb{F}_\infty \otimes C([0, 1], B)}{I_J}.$$

Let $g_j \in C[0, 1]$ such that $g(s_j) = 1$ and $\text{supp } g_j \subset (t_j, t_{j+1})$. Then for $j \notin J$, g_j induces a continuous u.c.p. map $\hat{\pi}_j : A \otimes_{\min} B \rightarrow A \otimes_J C([0, 1], B)$ which maps $a \otimes b \in A \otimes_m B$ to $a \otimes bg_j$. For $j \in J$, there exists an evaluation map $\text{ev}_{1j} : C([0, 1], B) \rightarrow B$, which maps $f \otimes b$ to $f(s_j)b$. Then we have the following commuting diagram

$$\begin{array}{ccccc} C^*\mathbb{F}_\infty \otimes_{\max} B & \xrightarrow{\text{id} \otimes \pi} & C^*\mathbb{F}_\infty \otimes_{\max} C([0, 1], B) & \xrightarrow{\text{id} \otimes \text{ev}_{1j}} & C^*\mathbb{F}_\infty \otimes_{\max} B \\ \downarrow & & \downarrow & & \downarrow \\ A \otimes_{\min} B & \xrightarrow{\hat{\pi}_j} & A \otimes_J C([0, 1], B) & \longrightarrow & A \otimes_{\max} B \end{array}$$

which contradicts the assumption. □

The norm construction in Theorem 5.3.1 can be applied in the case of von Neumann algebra, where the infinite copies of min-tensor product can be replaced by von Neumann tensor product. we define the J -norm on $A \otimes N^{\otimes \mathbb{N}}$ for $J \subset \mathbb{N}$, as a quotient from $C^*\mathbb{F}_\infty \otimes_{\text{nor}} N^{\otimes \mathbb{N}}$ by I_J . The right-normality will take care of all issues of continuity. Therefore we have the following proposition:

Proposition 5.3.4. *For a unital C^* -algebra A and a von Neumann algebra N such that $A \otimes_{\min} N \neq A \otimes_{\text{nor}} N$, there are 2^{\aleph_0} distinct norms on $A \otimes N^{\otimes \mathbb{N}}$.*

5.4 Constructions of norms from subalgebras and quotients

Now we would like to construct many norms based on the property of being QWEP.

Remark 5.4.1. Notice that A being QWEP is equivalent to say that A is *r.w.i.* in $\mathbb{B}(\mathcal{H})^{**}$. Therefore if there exists a C^* -algebra A , which is not QWEP, we could use the non-*r.w.i.* pair $A \subset \mathbb{B}(\mathcal{H})^{**}$ to construct the norms as in Section 2. On the other hand, we could directly construct the norms by other equivalent conditions, and we hope the new construction can be applied to other cases.

Recall that if a C^* -algebra is QWEP if and only if for any u.c.p. $u : C^*\mathbb{F}_\infty \rightarrow A$, we have the map $u \otimes \text{id} : C^*\mathbb{F}_\infty \otimes_{\min} A^{\text{op}} \rightarrow A \otimes_{\max} A^{\text{op}}$ is continuous. Define the q -norm on $A \otimes A^{\text{op}}$ to be the

quotient norm

$$A \otimes_q A^{\text{op}} = \frac{C^*\mathbb{F}_\infty \otimes_{\min} A^{\text{op}}}{I \otimes_{\min} A^{\text{op}}}.$$

Then the condition is equivalent to say that $A \otimes_q A^{\text{op}} = A \otimes_{\max} A^{\text{op}}$.

Theorem 5.4.2. *Suppose that there exists a C^* -algebra A which is not QWEP. Then there are 2^{\aleph_0} number of distinct norms on $A \otimes (A^{\text{op}})^{\otimes \mathbb{N}}$.*

Proof. Let $A^{\text{op}} \subset \mathbb{B}(\mathcal{H})$ be a faithful representation. For $J \subset \mathbb{N}$, let the C^* -algebra D_J be the min tensor product of D_j for $j \in \mathbb{N}$

$$D_j = \begin{cases} A & \text{if } j \in J; \\ \mathbb{B}(\mathcal{H}) & \text{if } j \notin J. \end{cases}$$

Define the J -norm on $C^*\mathbb{F}_\infty \otimes (A^{\text{op}})^{\otimes \mathbb{N}}$ to be the closure $C^*\mathbb{F}_\infty \otimes_J (A^{\text{op}})^{\otimes \mathbb{N}} \subset C^*\mathbb{F}_\infty \otimes_{\max} D_J$, and the J -norm on $A \otimes (A^{\text{op}})^{\otimes \mathbb{N}}$ to be the quotient norm

$$A \otimes_J (A^{\text{op}})^{\otimes \mathbb{N}} = \frac{C^*\mathbb{F}_\infty \otimes_J (A^{\text{op}})^{\otimes \mathbb{N}}}{I \otimes_J (A^{\text{op}})^{\otimes \mathbb{N}}}.$$

Suppose for $J \neq J'$, we have $A \otimes_J (A^{\text{op}})^{\otimes \mathbb{N}} = A \otimes_{J'} (A^{\text{op}})^{\otimes \mathbb{N}}$. Let $j \in J \setminus J'$, consider the j -th component on both sides of the tensor algebra.

On the left hand side, since $C^*\mathbb{F}_\infty \otimes_J (A^{\text{op}})^{\otimes \mathbb{N}} \subset C^*\mathbb{F}_\infty \otimes_{\max} D_J$, and $C^*\mathbb{F}_\infty \otimes_{\max} A^{\text{op}}$ is on the j -th component, we have that

$$A \otimes_{\max} A^{\text{op}} = \frac{C^*\mathbb{F}_\infty \otimes_{\max} A^{\text{op}}}{I \otimes_{\max} A^{\text{op}}} \subset \frac{C^*\mathbb{F}_\infty \otimes_J (A^{\text{op}})^{\otimes \mathbb{N}}}{I \otimes_J (A^{\text{op}})^{\otimes \mathbb{N}}} = A \otimes_J (A^{\text{op}})^{\otimes \mathbb{N}}.$$

On the right hand side, since $C^*\mathbb{F}_\infty \otimes_{\min} \mathbb{B}(\mathcal{H}) = C^*\mathbb{F}_\infty \otimes_{\max} \mathbb{B}(\mathcal{H})$ by [Ki93], we have that $C^*\mathbb{F}_\infty \otimes_{\min} A^{\text{op}} \subset C^*\mathbb{F}_\infty \otimes_{\min} \mathbb{B}(\mathcal{H}) = C^*\mathbb{F}_\infty \otimes_{\max} \mathbb{B}(\mathcal{H}) \subset C^*\mathbb{F}_\infty \otimes_{\max} D_{J'}$. Therefore we have the quotient norm

$$A \otimes_q A^{\text{op}} = \frac{C^*\mathbb{F}_\infty \otimes_{\min} A^{\text{op}}}{I \otimes_{\min} A^{\text{op}}} \subset \frac{C^*\mathbb{F}_\infty \otimes_{J'} (A^{\text{op}})^{\otimes \mathbb{N}}}{I \otimes_{J'} (A^{\text{op}})^{\otimes \mathbb{N}}} = A \otimes_{J'} (A^{\text{op}})^{\otimes \mathbb{N}}.$$

Hence we have $A \otimes_q A^{\text{op}} = A \otimes_{\max} A^{\text{op}}$ on the j -th component, which contradicts the assumption that A is not QWEP. \square

To complete the picture, we also have similar results for the constructions from infinite copies of

A^{op} . Since the proofs are similar, this will be left to the readers' interest.

Proposition 5.4.3. *Suppose there exists a C^* -algebra A , which is not $QWEP$. Then there are 2^{\aleph_0} number of distinct norms on the following algebras:*

1. $A \otimes (*_{\mathbb{N}}(A^{\text{op}}, \varphi));$
2. $A \otimes c_0(A^{\text{op}}).$
3. $A \otimes C([0, 1], A^{\text{op}}).$

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